Starting a Fire with Twigs: Influence of Encapsulation Relations on Bottom-up Dynamics on Hypergraphs

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NetSci 2023

MATHEMATICAL INSTITUTE
UNIVERSITY OF OXFORD

In collaboration with Prof. Renaud Lambiotte

Multi-way/higher-order interactions

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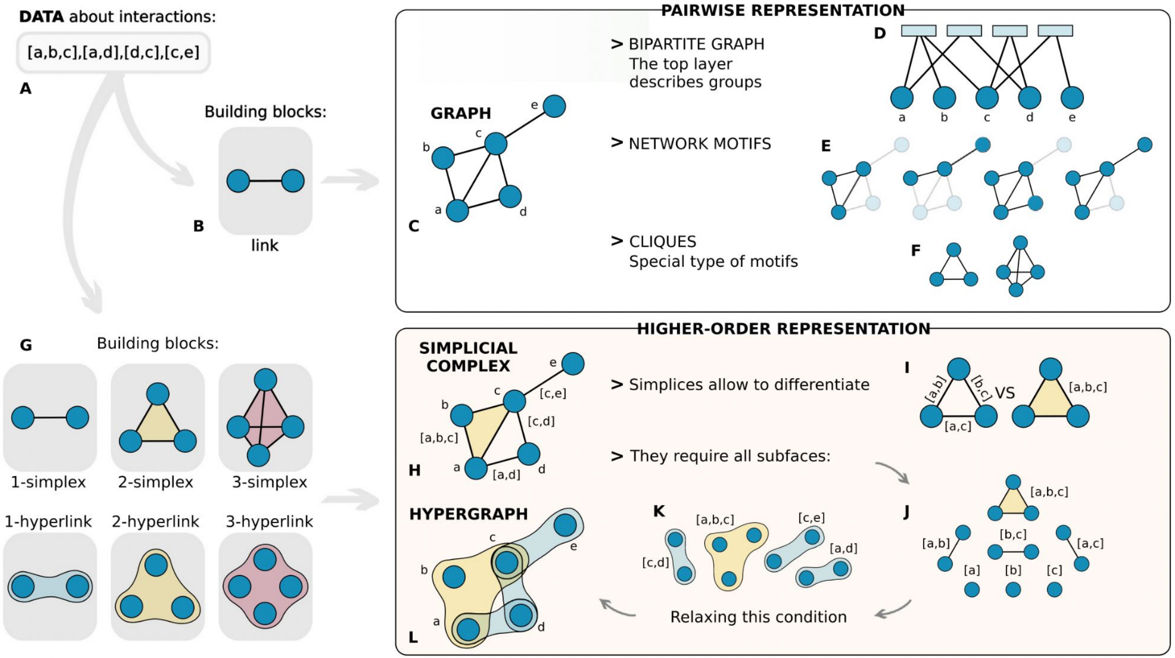

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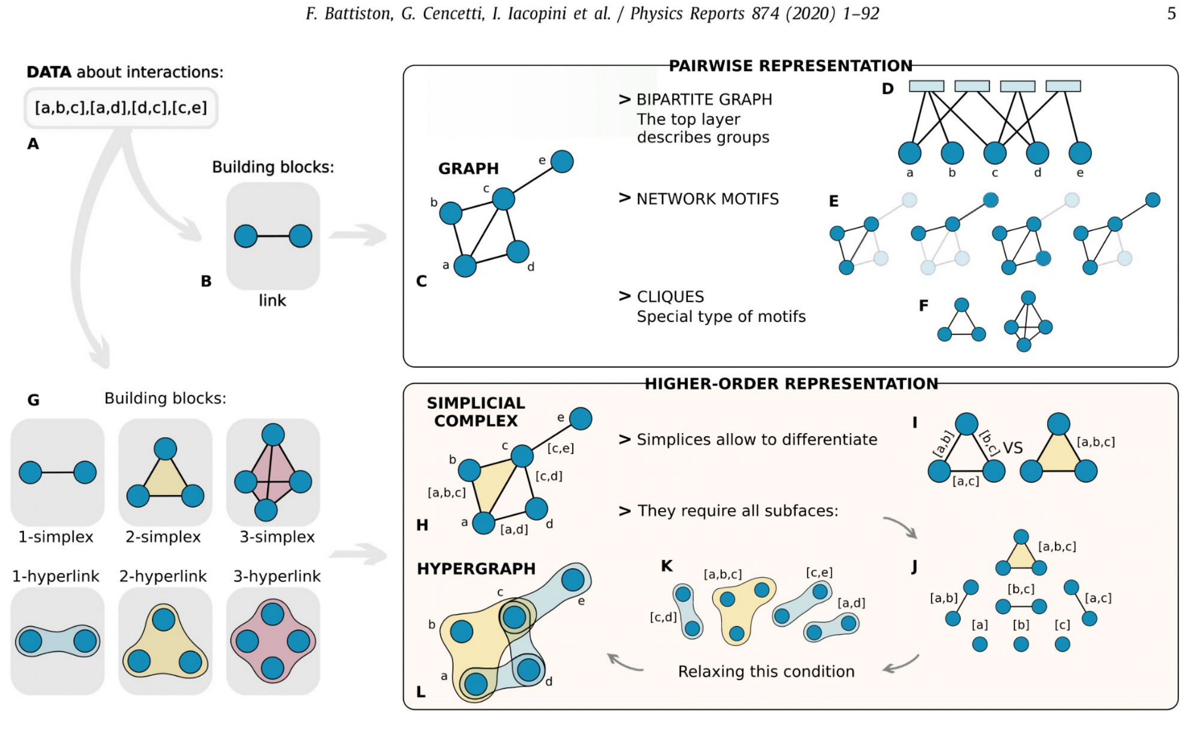

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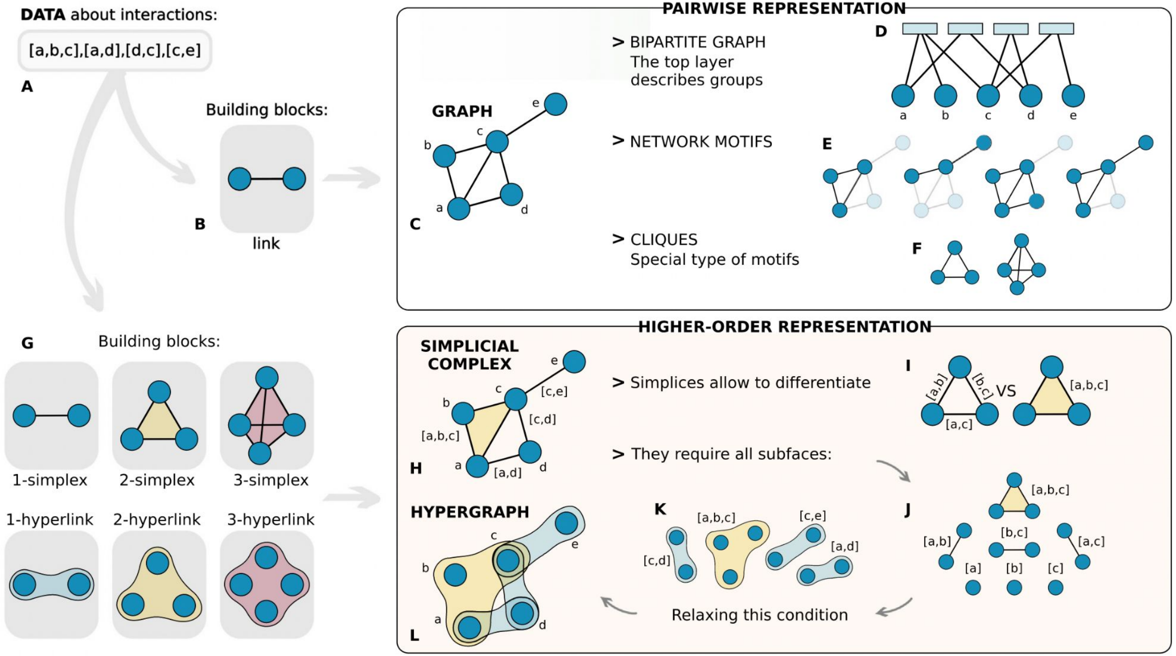

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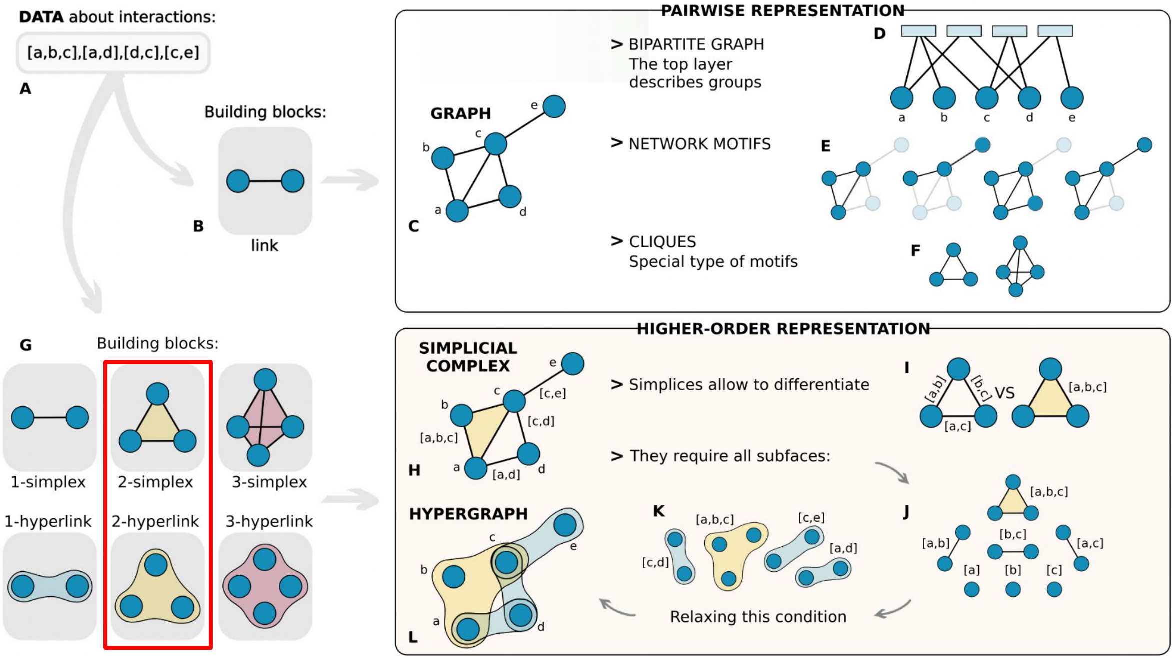

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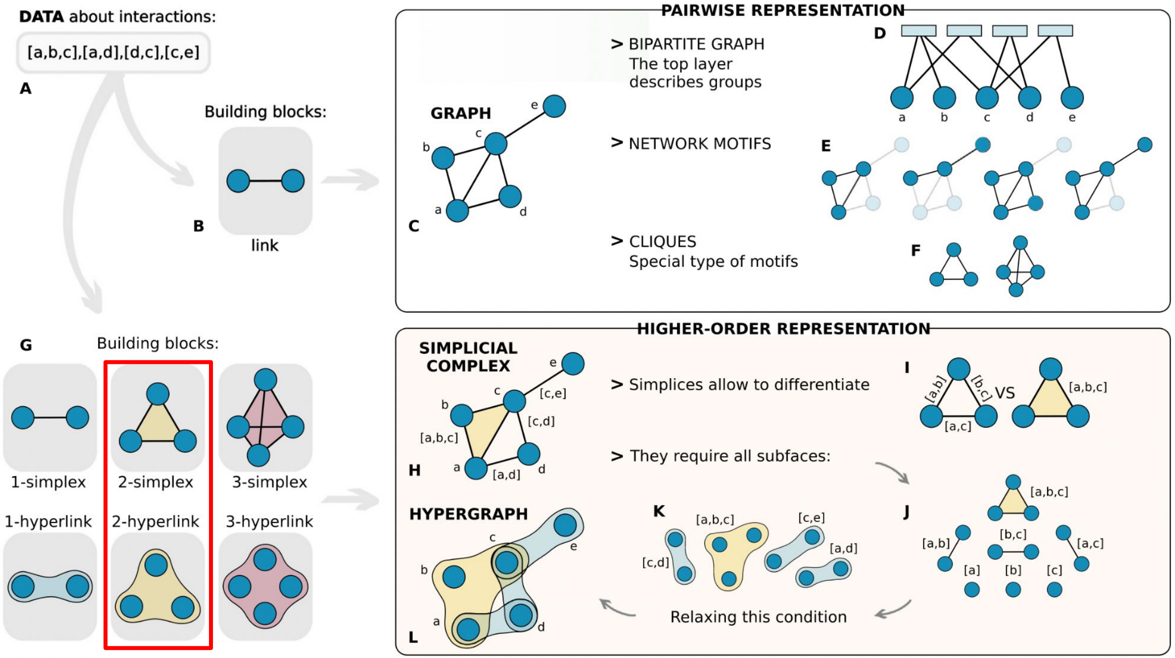

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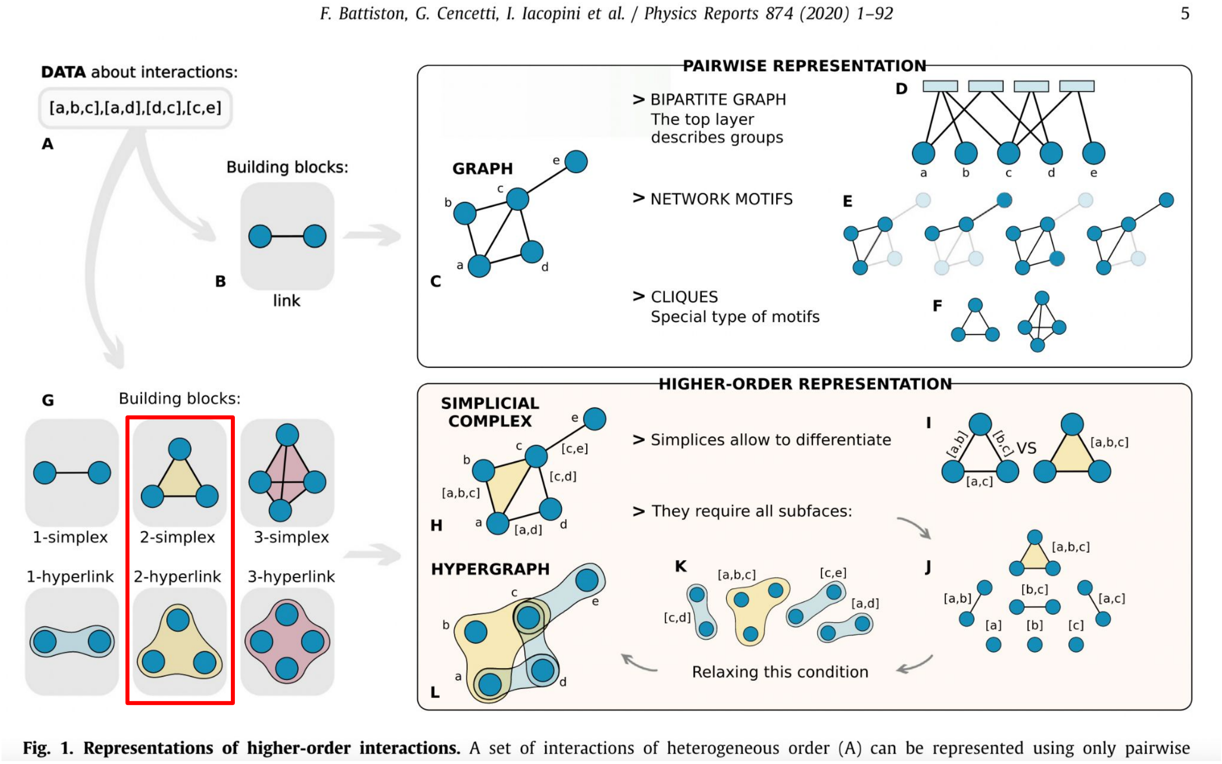
Our goal: Characterize hypergraph structure based on ***encapsulation***, then study a contagion process driven by encapsulation where activation occurs **on hyperedges**.



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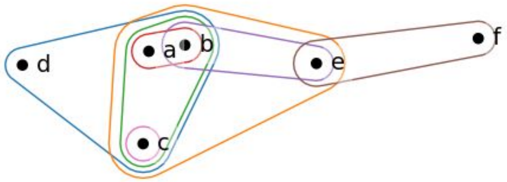
Encapsulation, Overlap, and Line Graphs of Hypergraphs

Hypergraph (Euler diagram)



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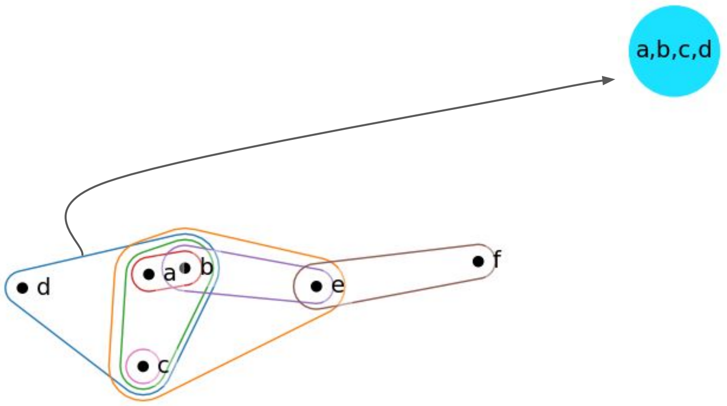

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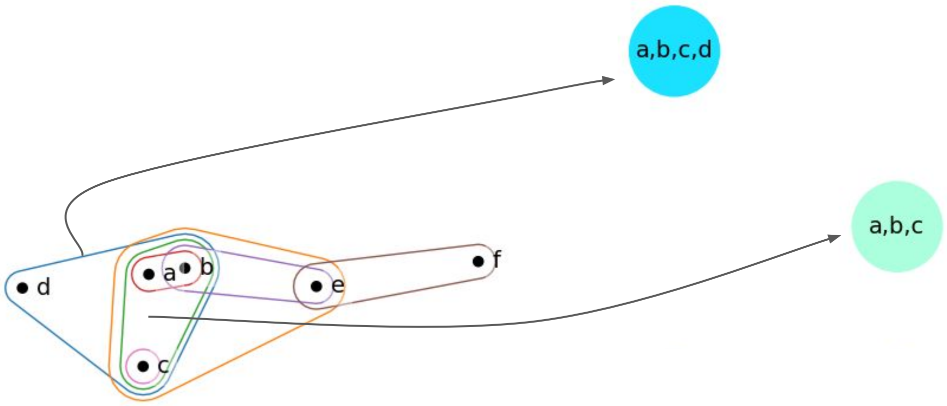

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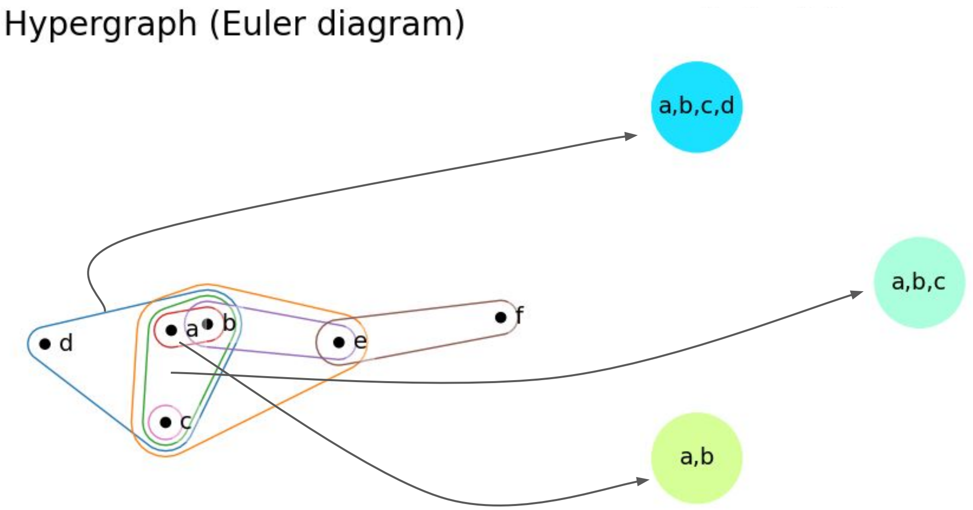

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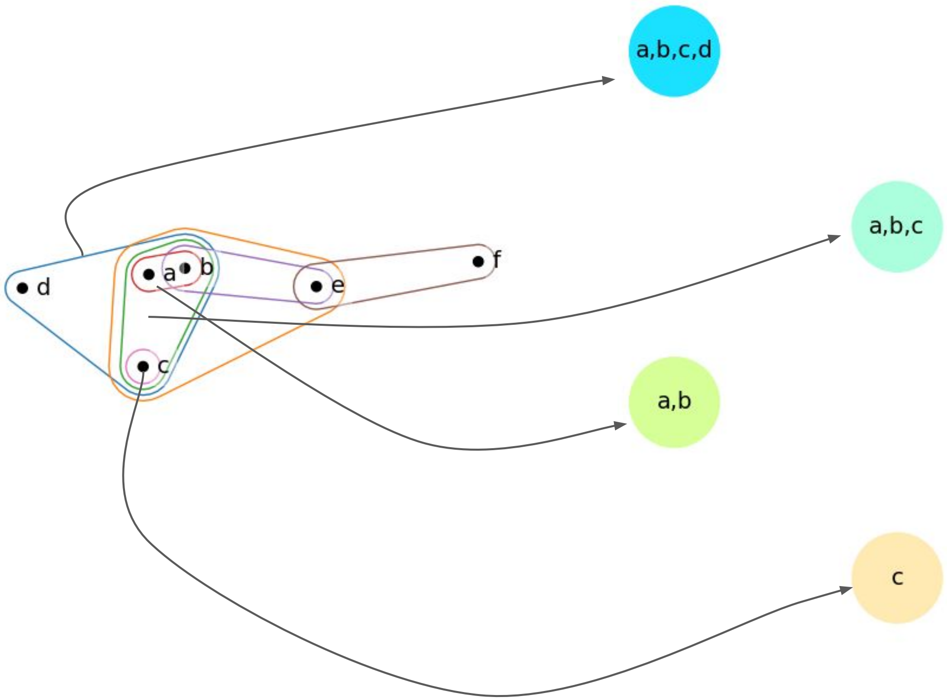

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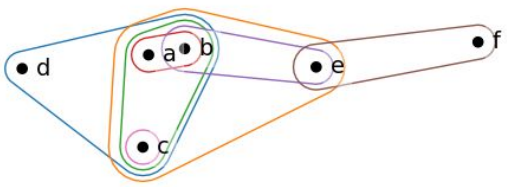

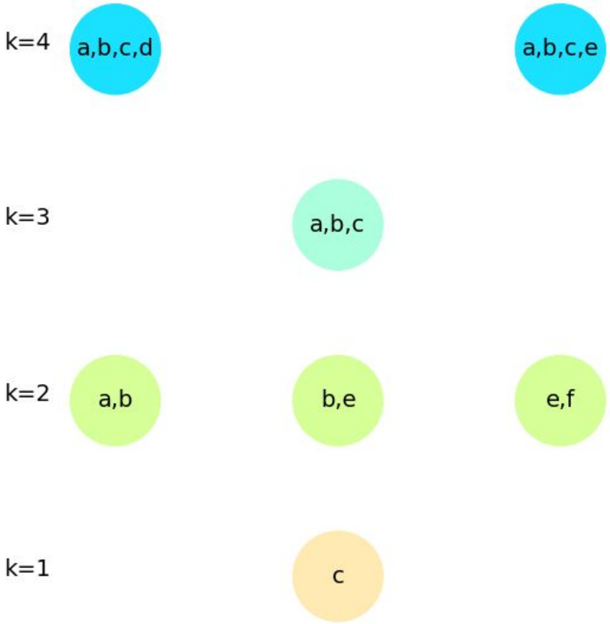

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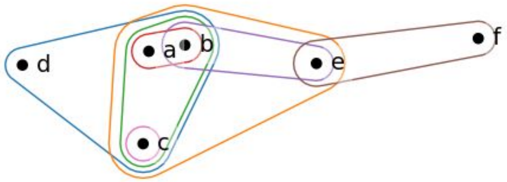

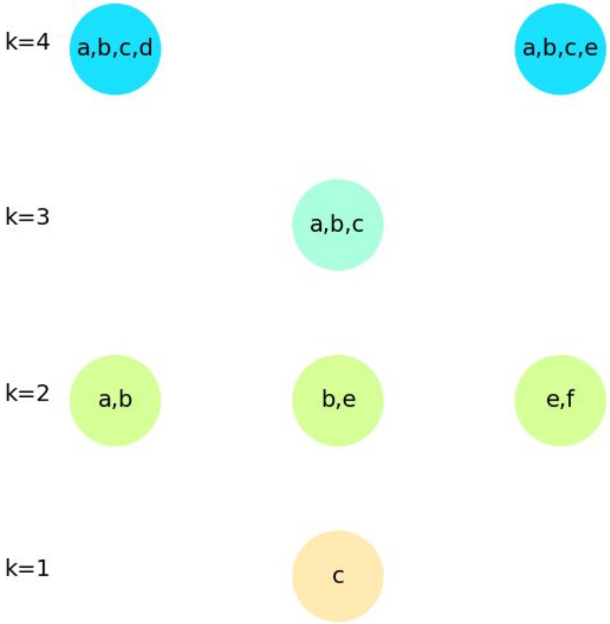

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Encapsulation, Overlap, and Line Graphs of Hypergraphs

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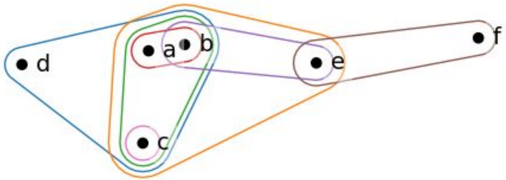

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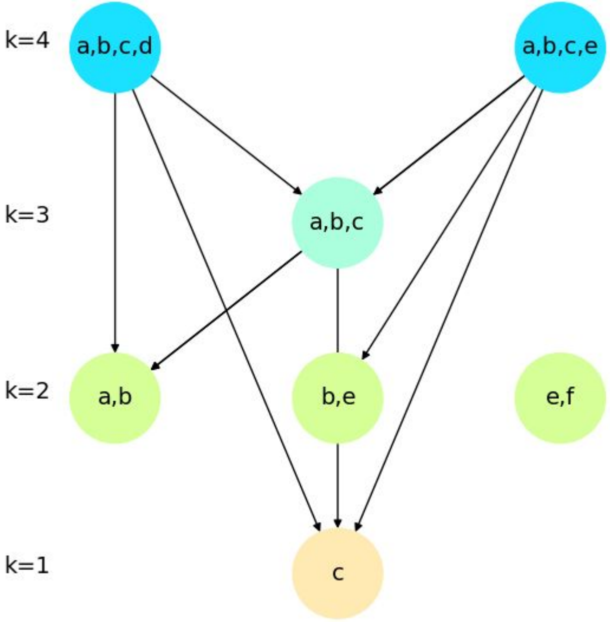

Encapsulation DAG

Encapsulation, Overlap, and Line Graphs of Hypergraphs

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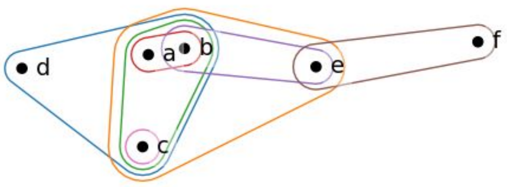

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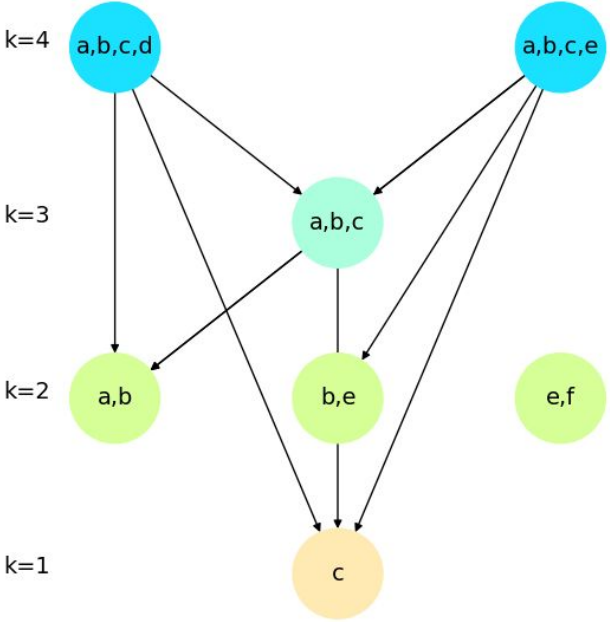
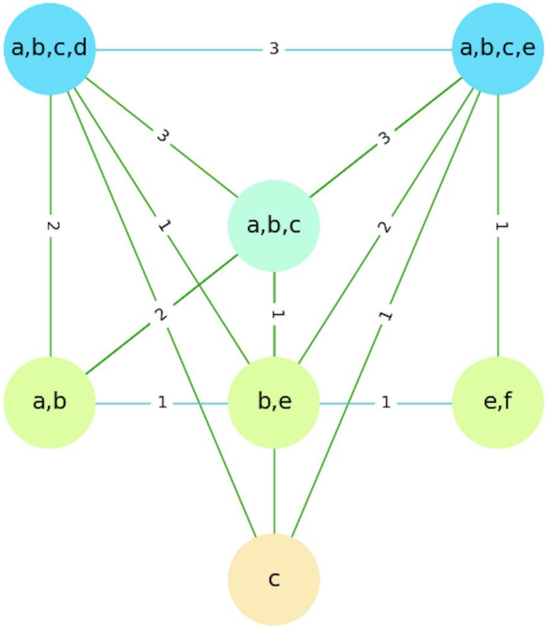


Encapsulation DAG

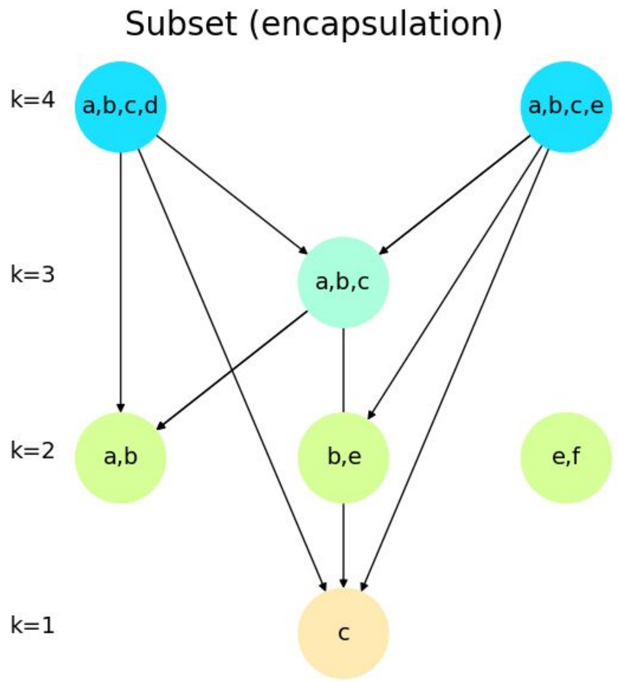
Intersection (overlap)



Overlap Line Graph

Why encapsulation DAGs?

Number of edges → total number of encapsulation relationships



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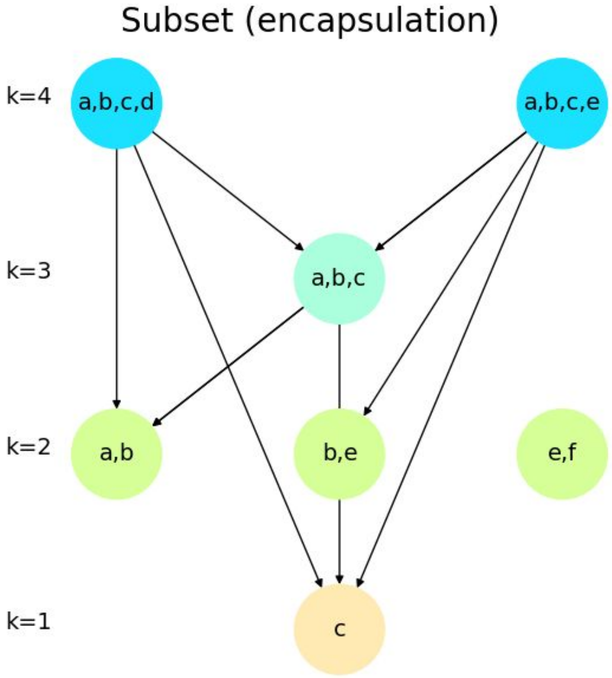

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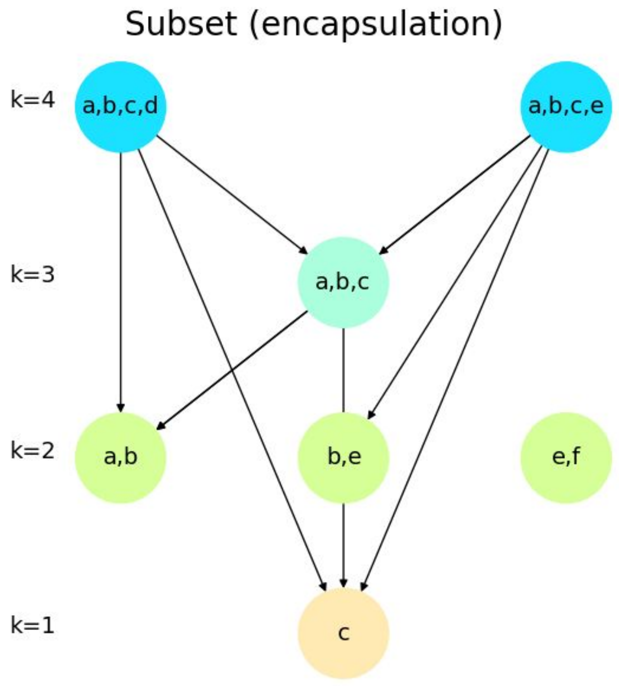

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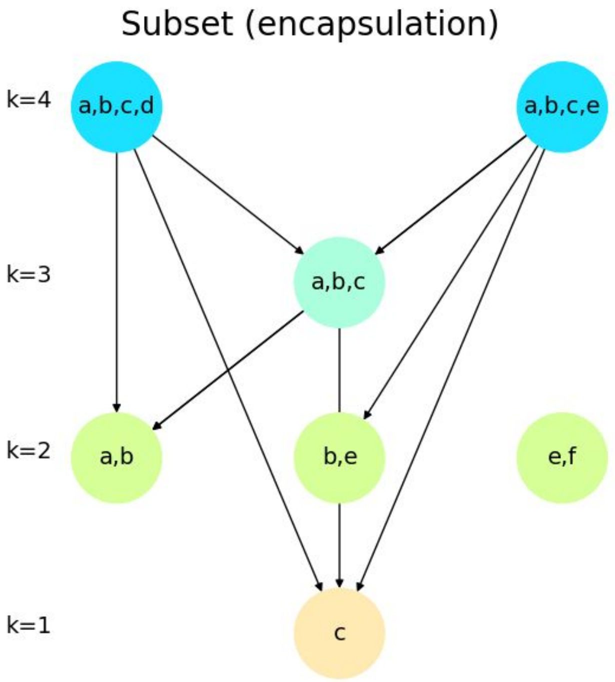

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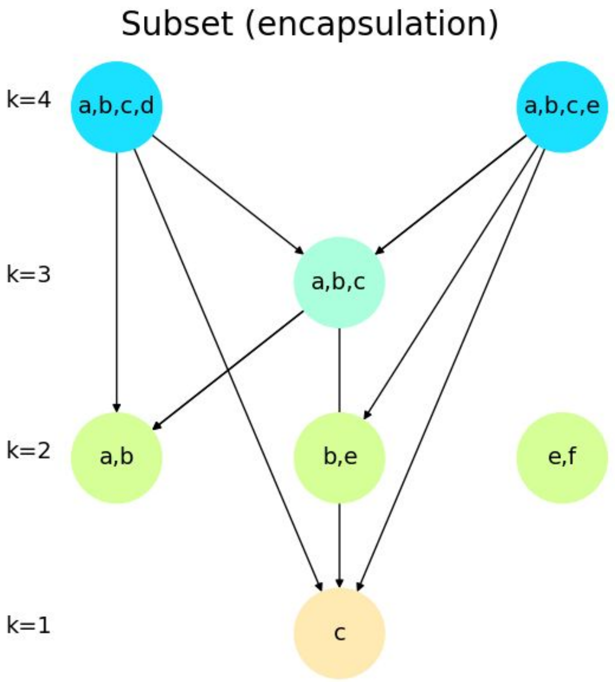

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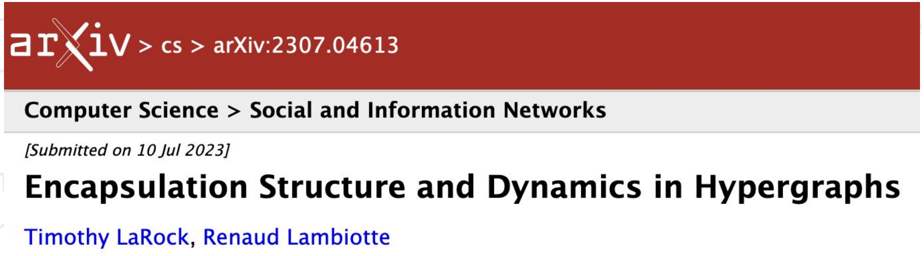
arXiv > cs > arXiv:2307.04613

Computer Science > Social and Information Networks

[Submitted on 10 Jul 2023]

Encapsulation Structure and Dynamics in Hypergraphs

Timothy LaRock, Renaud Lambiotte

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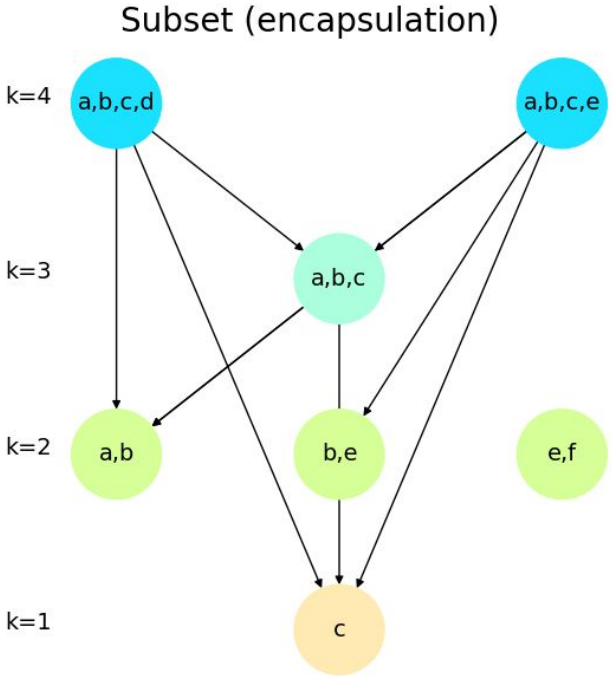


Dynamics on hypergraphs

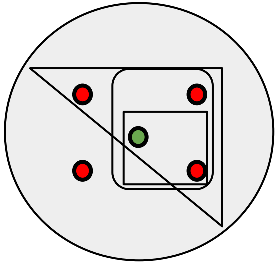
Node-based Threshold
Dynamics

A **node** becomes active if it participates
in a hyperedge where more than a
threshold of **nodes** become active.

Dynamics on hypergraphs

Node-based Threshold Dynamics

t



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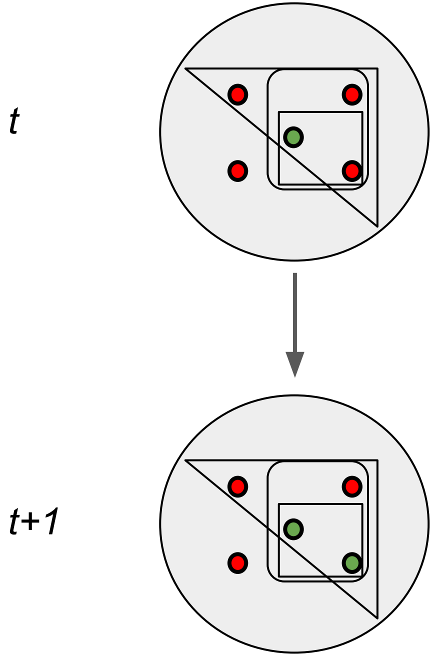

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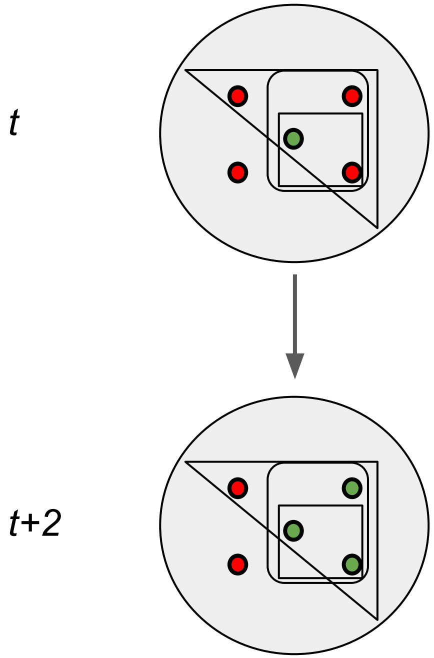

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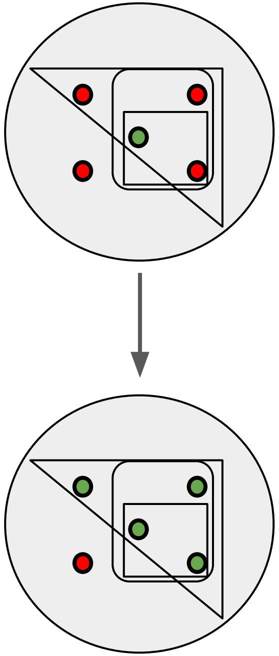

t+3

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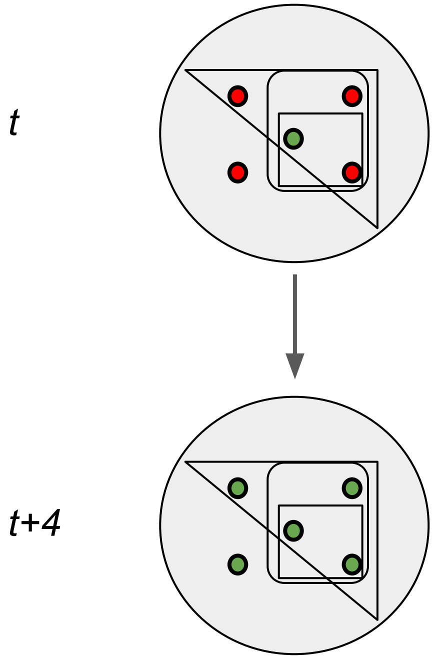

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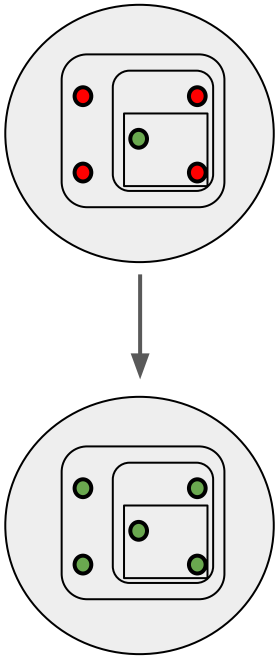

t+4

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Encapsulation Dynamics

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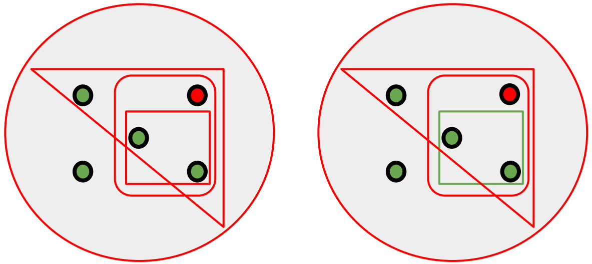

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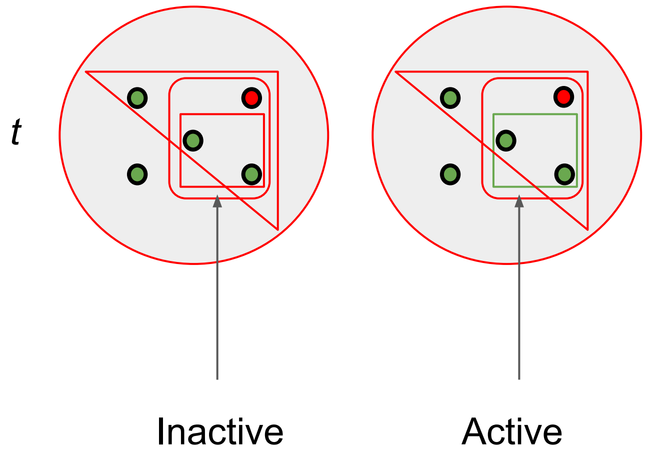
Dynamics on hypergraphs

Nodes **and** edges in binary state, active or inactive

Encapsulation Dynamics



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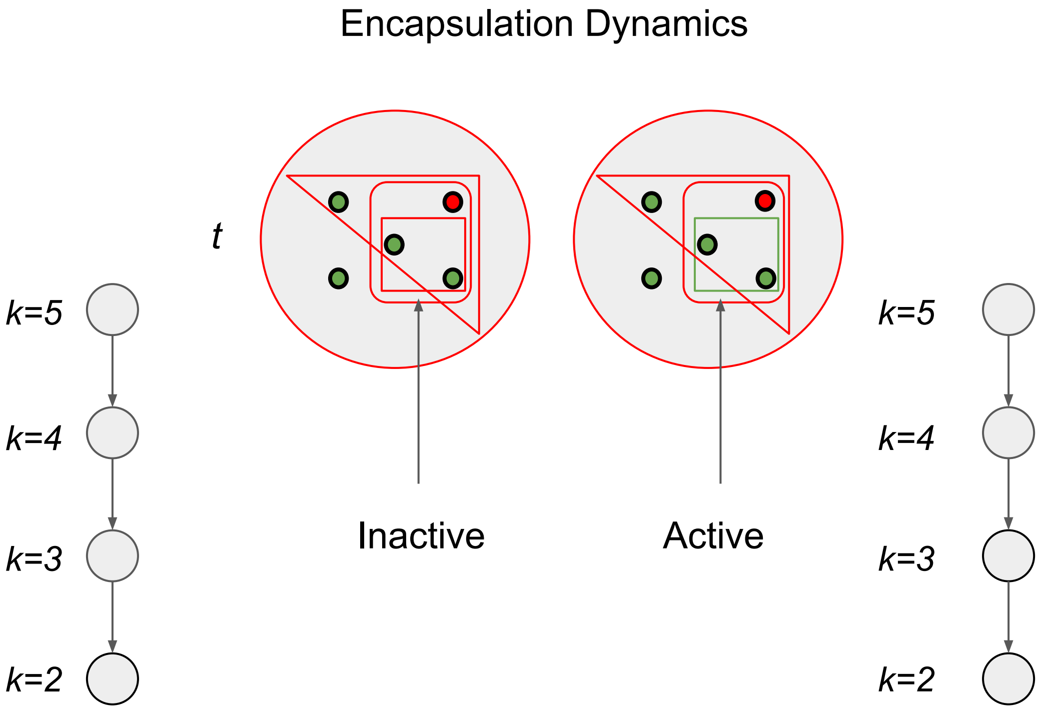

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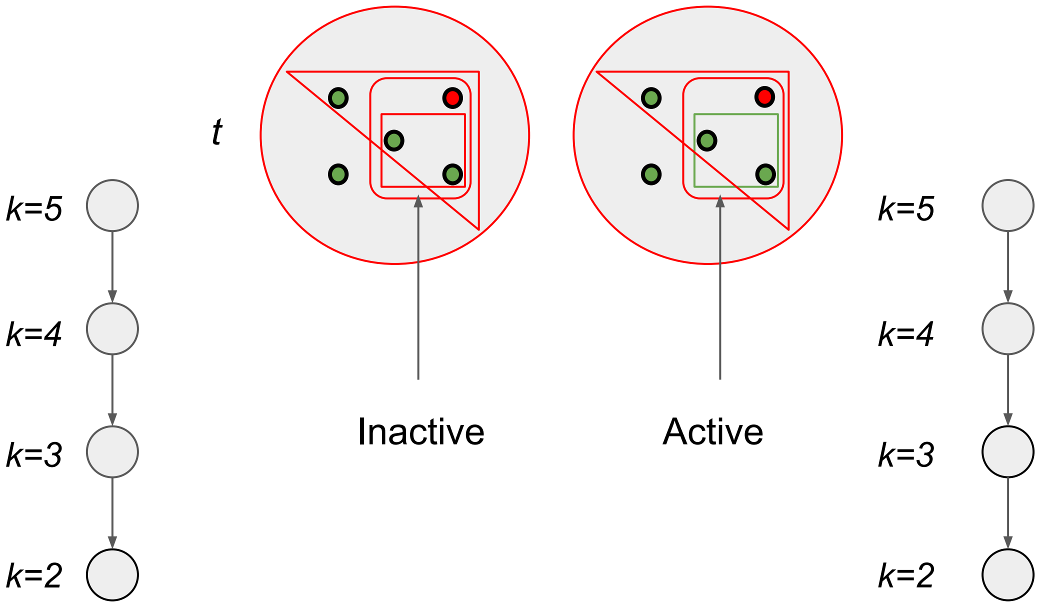

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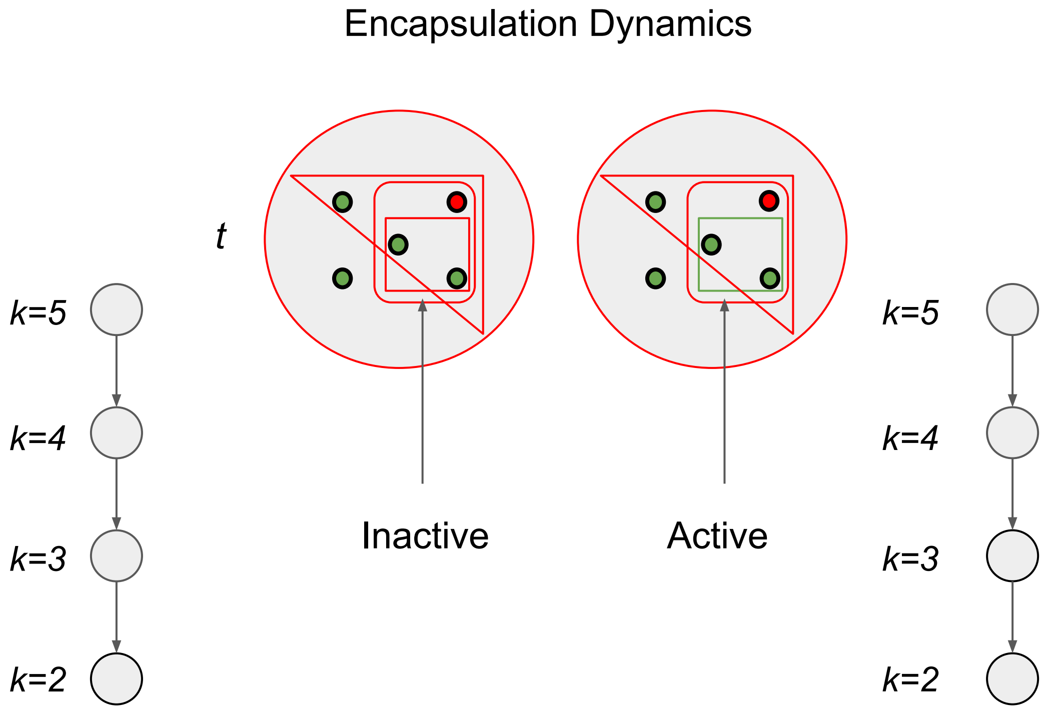

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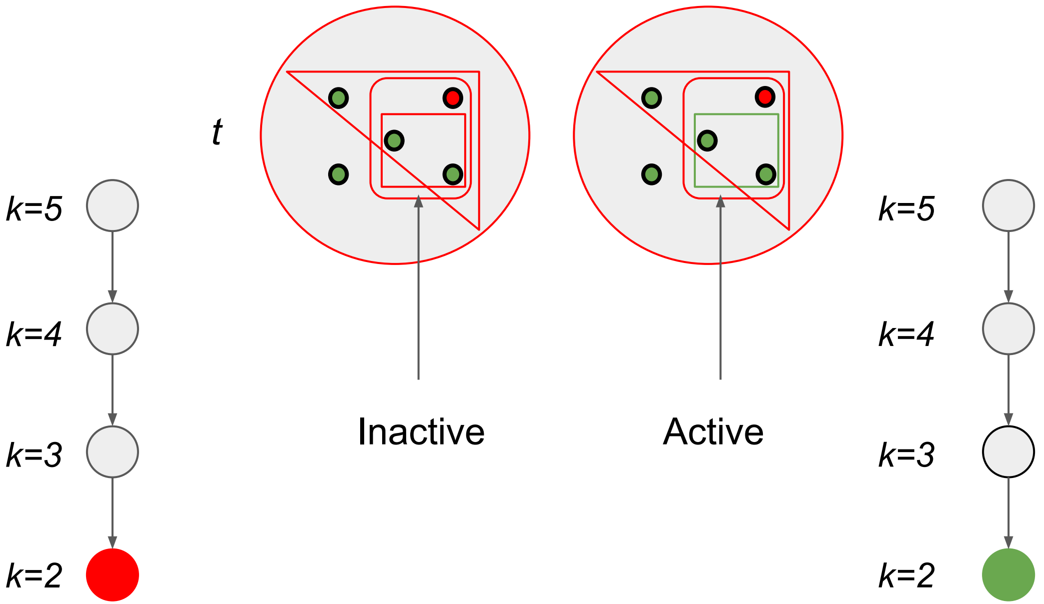

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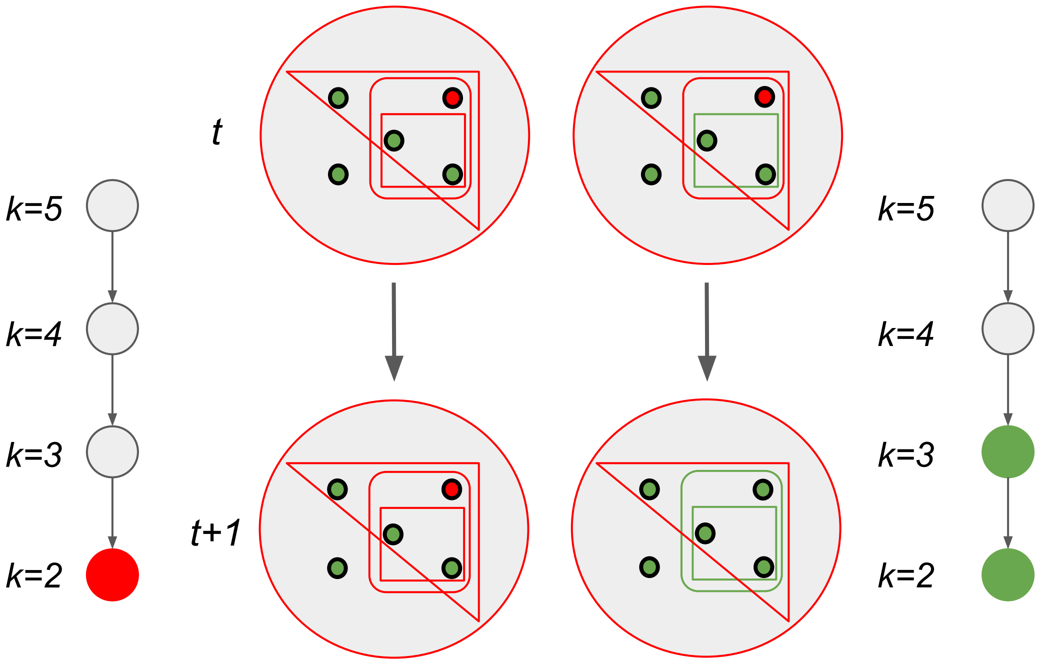

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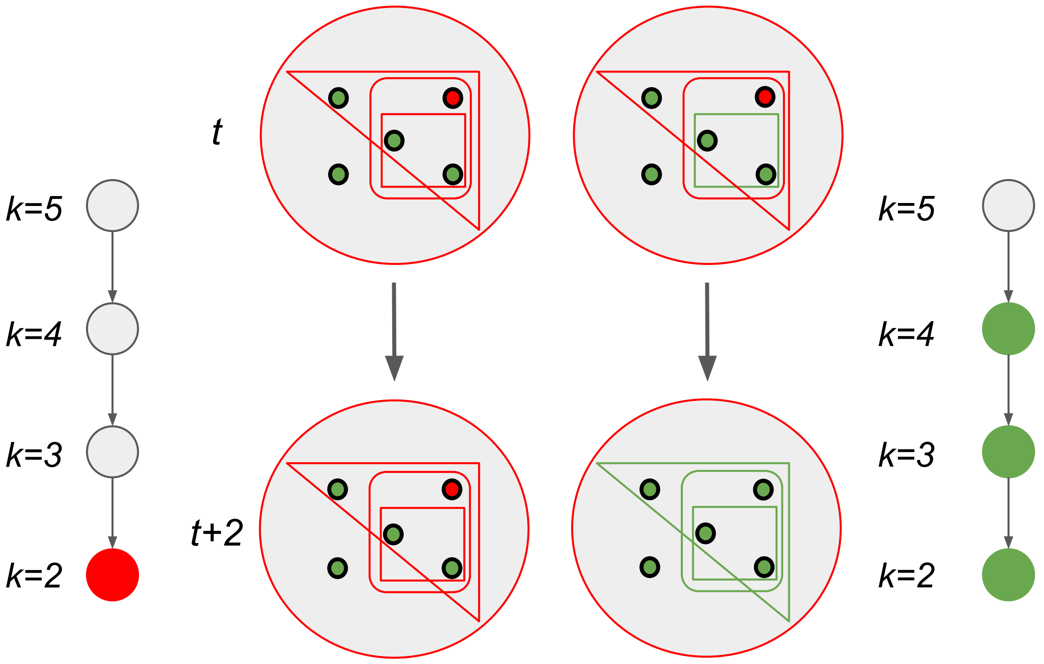

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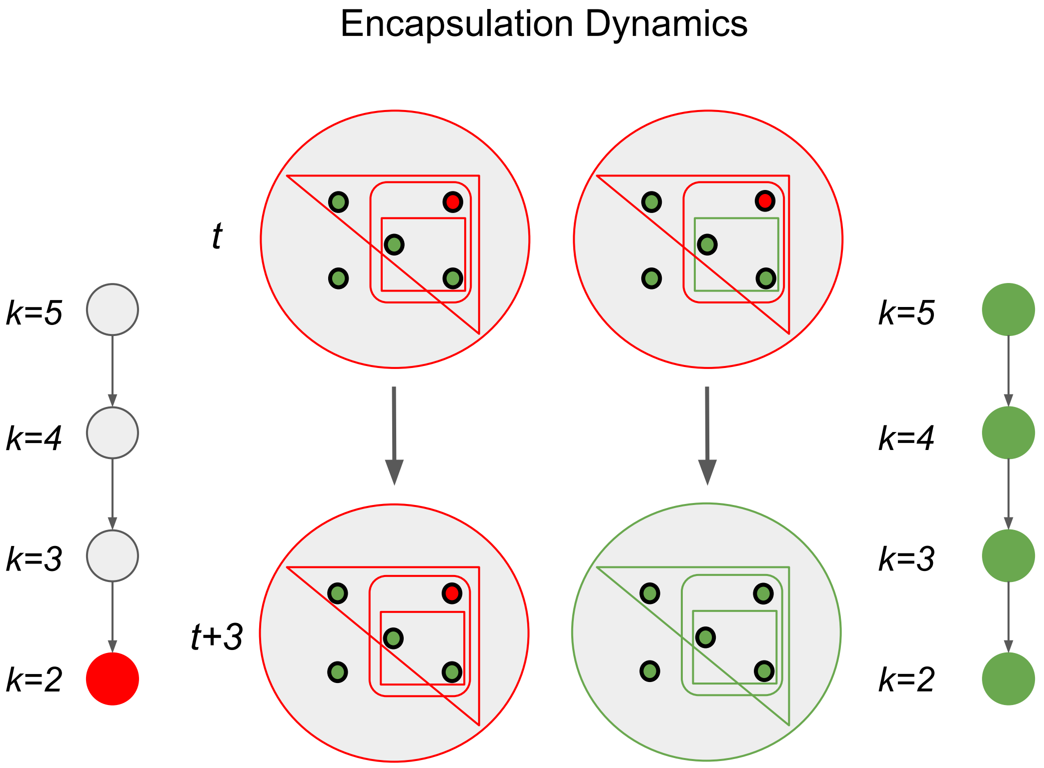

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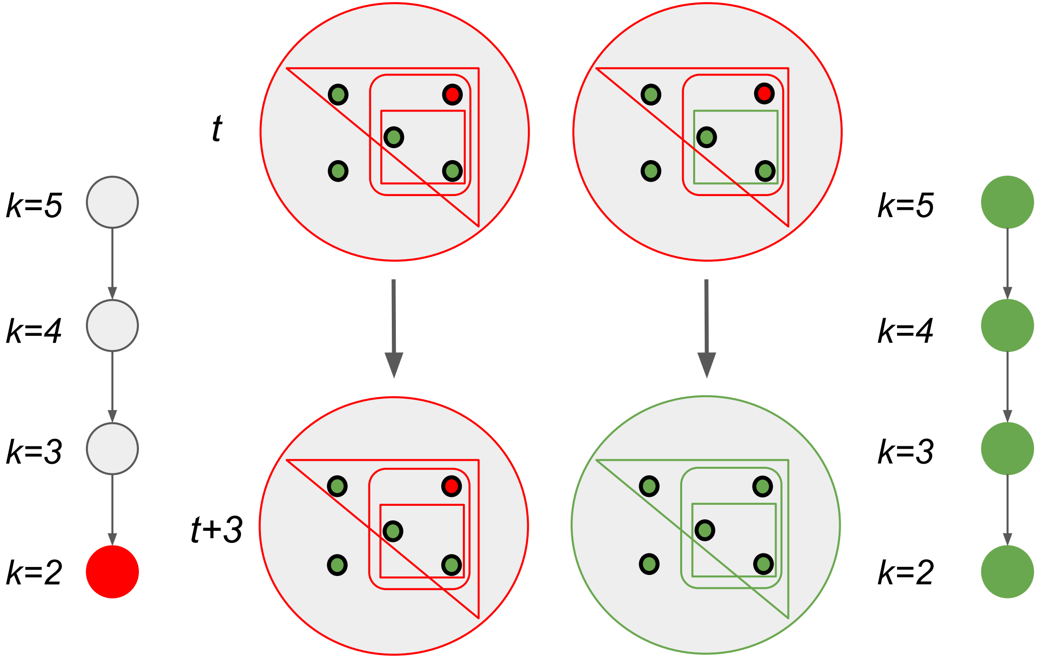
(Campfire) Dynamics on hypergraphs

Analogy to lighting a campfire: the smallest fuel must be lit before the logs can catch on fire!

Image: https://www.pelican.com/us/en/discover/pelican-flyer/post/how-to-start-a-campfire/

Correspond to a type of **coordinated behavior** where nodes not only share goals/opinion/information, but coordinate to pass to other groups they are embedded within.

Encapsulation Dynamics



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Random Nested Hypergraph Model

Contagion dynamics on hypergraphs with nested hyperedges

Jihye Kim,[1] Deok-Sun Lee,[2,*] and Kwang-Il Goh[1,†]

arXiv:2303.00224v1

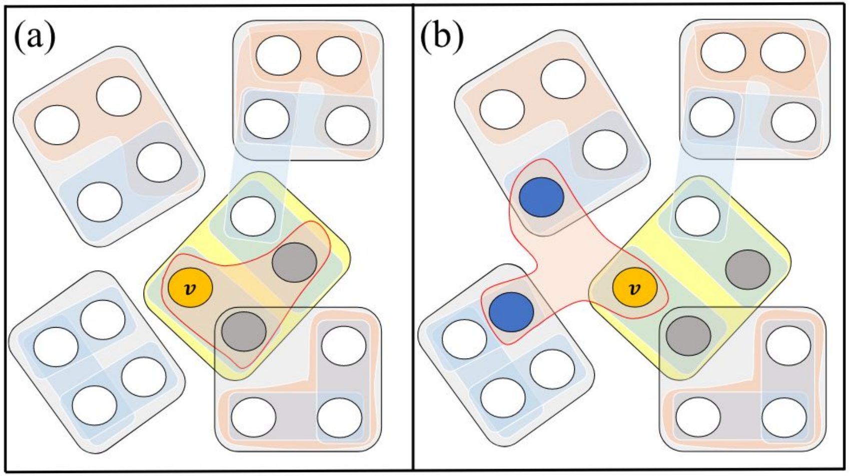
Idea: Start from a fully encapsulated hypergraph (simplicial complex), then selectively rewire hyperedges to destroy encapsulation relationships

Parameters:

- N: Number of nodes
- s_m: Maximum size hyperedge
- H_s: Number of hyperedges of size s_m
- ϵ_s: 1 minus probability of rewiring hyperedge of size s

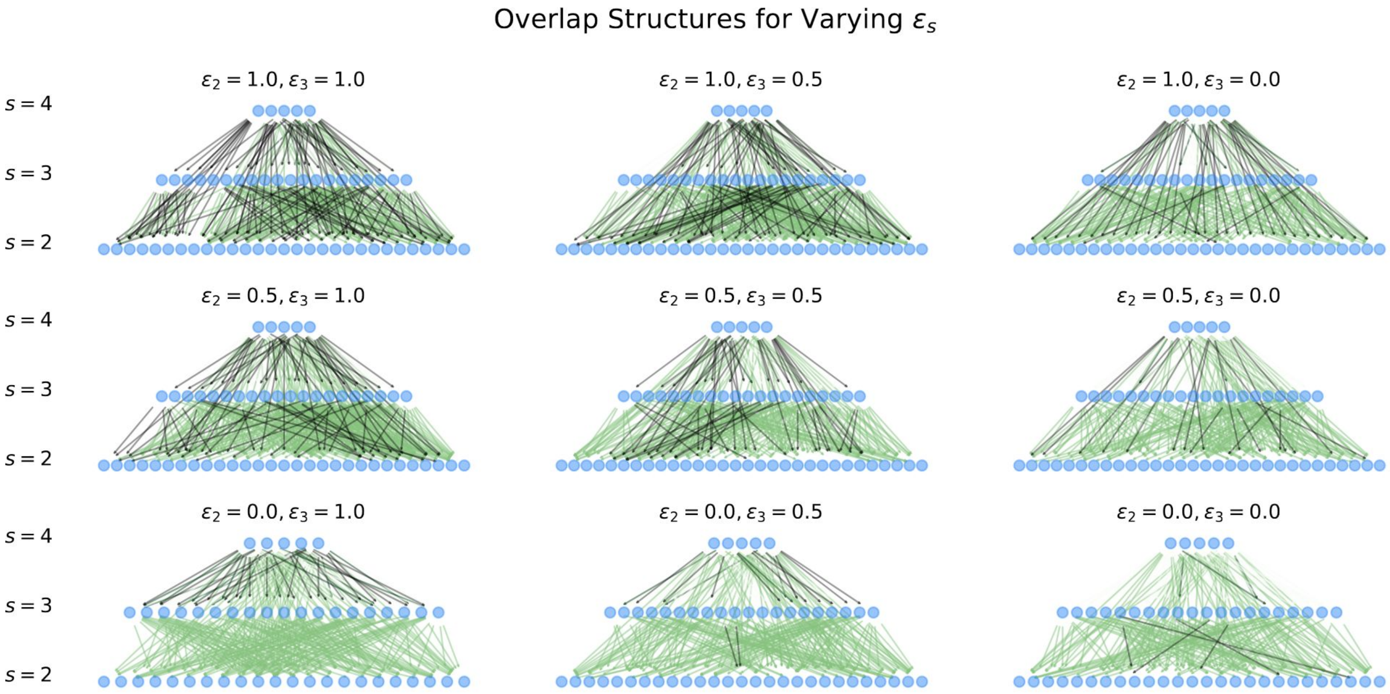
Procedure:

1. Generate random hyperedges of size s_m and all of their subhyperedges (power set)
2. For each hyperedge of size $s < s_m$, rewire with probability ϵ_s



Rewiring works by choosing a pivot node to keep, then randomizing other nodes by choosing nodes that are not in supersets of the original hyperedge.

Random Nested Hypergraph Model



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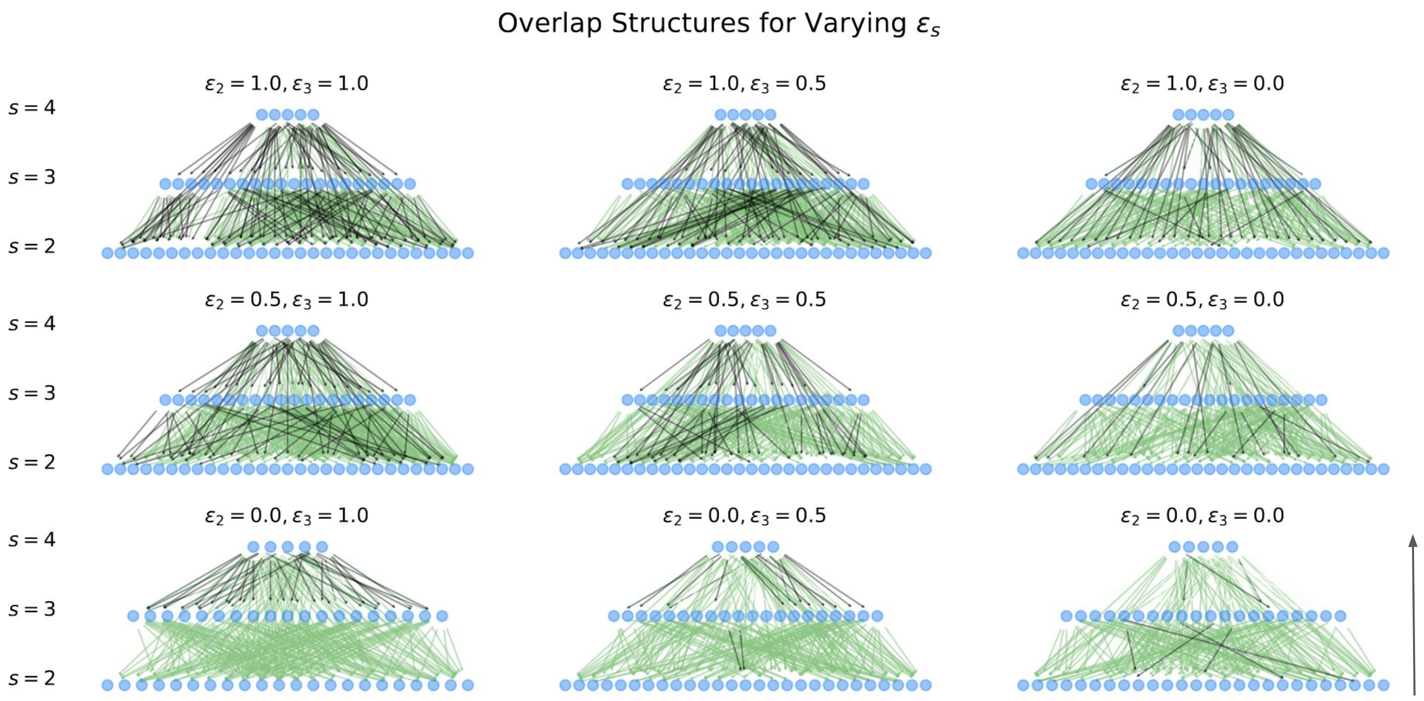
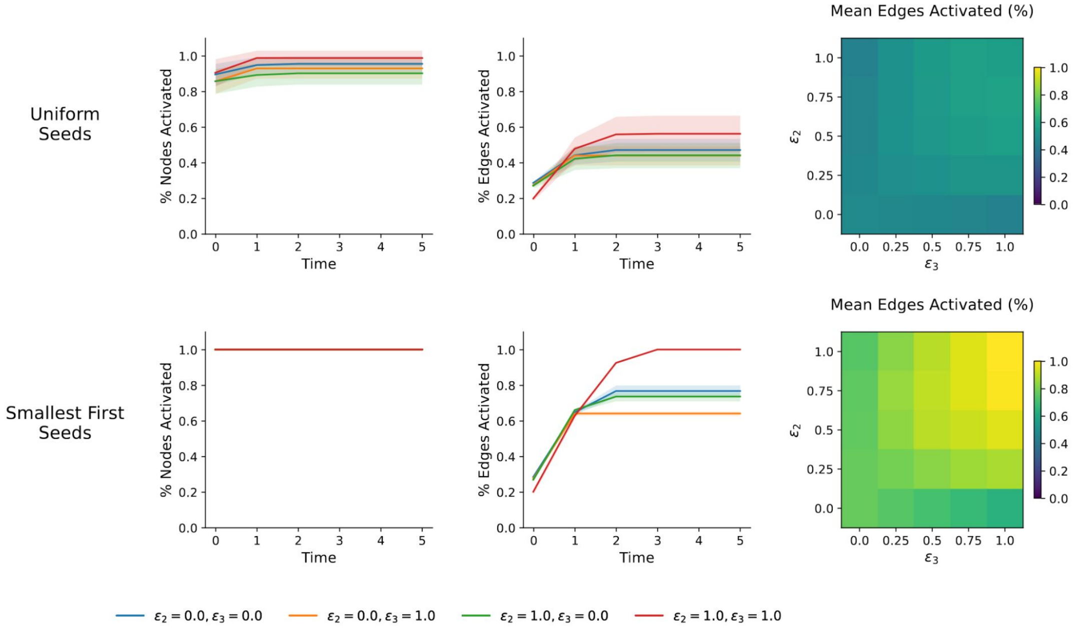


In strict encapsulation dynamics, **activation can only spread up the black edges!**

Simulation Results

Average results over 50 strict encapsulation dynamics simulations on 50 RNHM realizations



RNHM with N = 20, s_m = 4, H_s = 5, and varying ϵ_s, including individual nodes as hyperedges.

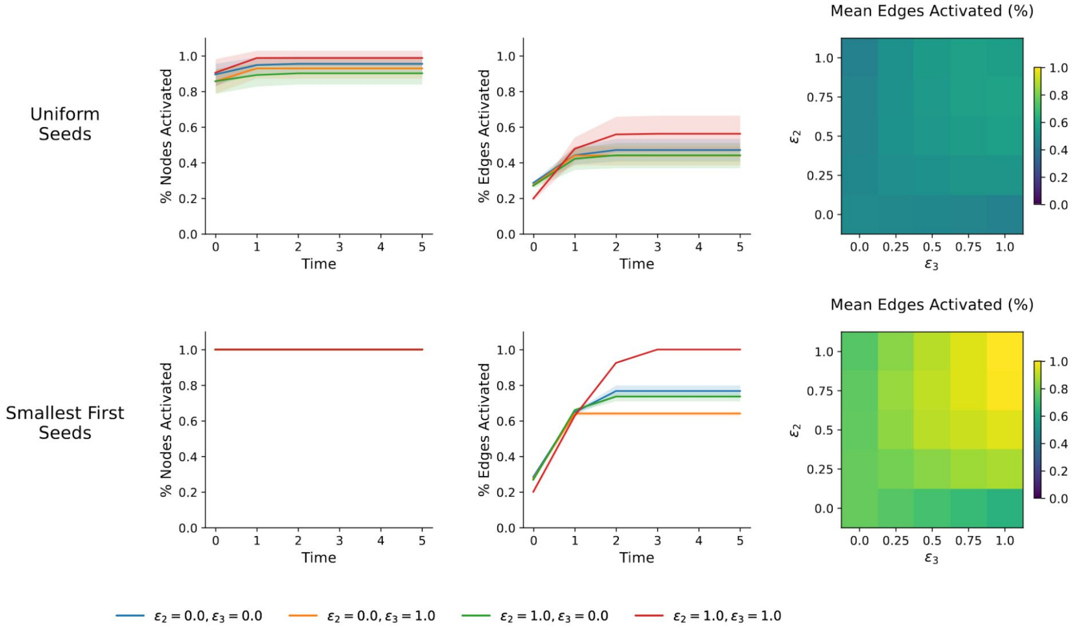
Number of seeds = N (number of nodes)

Simulation Results

Uniform seeding:

- Even with high proportion of nodes activated, only half of edes in the best case

Average results over 50 strict encapsulation dynamics simulations on 50 RNHM realizations



RNHM with N = 20, s_m = 4, H_s = 5, and varying ϵ_s, including individual nodes as hyperedges.

Number of seeds = N (number of nodes)

Simulation Results

Uniform seeding:

- Even with high proportion of nodes activated, only half of edes in the best case

Smallest first seeding:

- When no hyperedges are rewired, full hypergraph becomes activated (trivial but important)
- Even though nodes are activated by definition, all hyperedges do not become active. Key distinguishing feature from node-based threshold dynamics.

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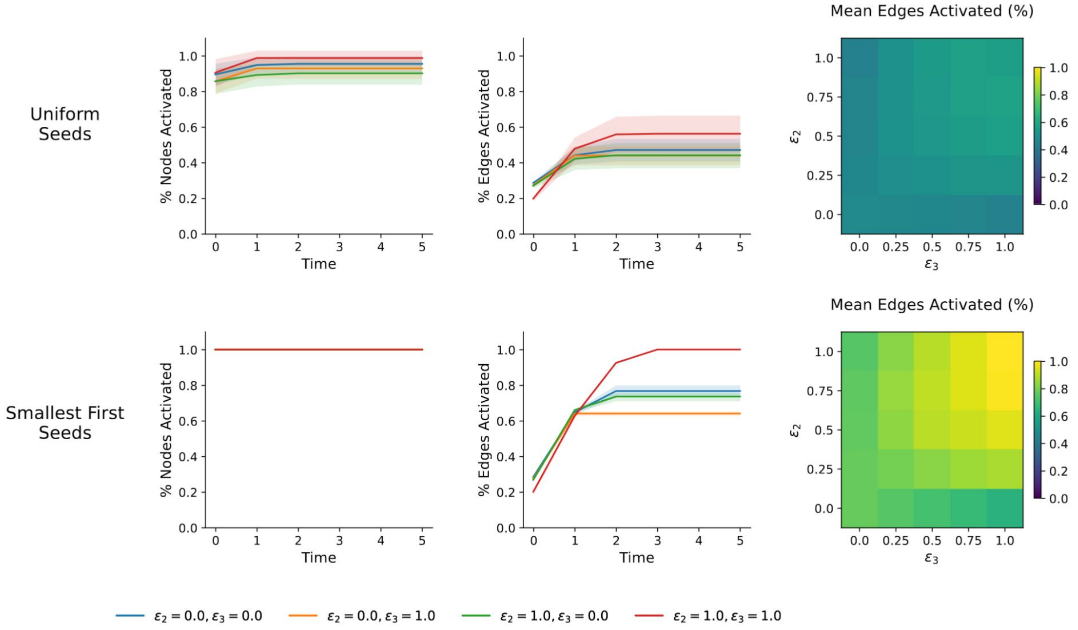

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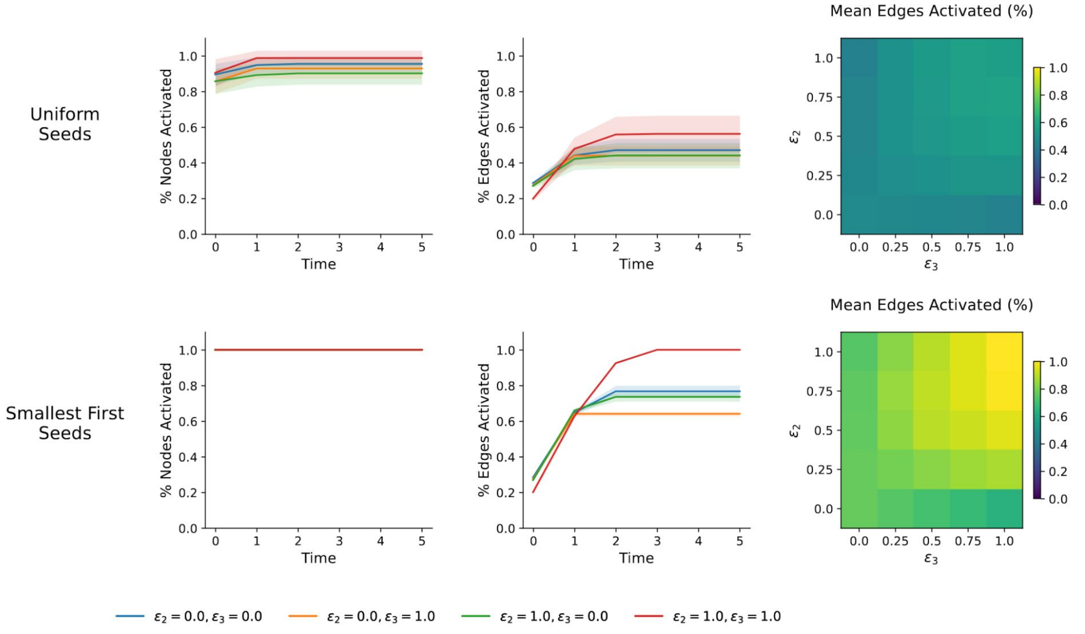
These dynamics correspond not just to node influence, but to **coordinated behavior**!

Average results over 50 strict encapsulation dynamics simulations on 50 RNHM realizations



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Number of seeds = N (number of nodes)

Thank you!

Contact
larock@maths.ox.ac.uk
https://www.tlarock.github.io

Code: @tlarock on GitHub

https://www.github.com/tlarock/encapsulation-dynamics

Also in the paper…

Measurements on empirical datasets

- DAG degrees, paths through encapsulation DAGs
- Comparison with randomization using a simple procedure that destroys encapsulation relationships

Comparison of strict vs. non-strict encapsulation dynamics on empirical hypergraphs, accounting for (non-)influence of individual nodes on pairwise interactions.

arXiv > cs > arXiv:2307.04613

Computer Science > Social and Information Networks

[Submitted on 10 Jul 2023]

Encapsulation Structure and Dynamics in Hypergraphs

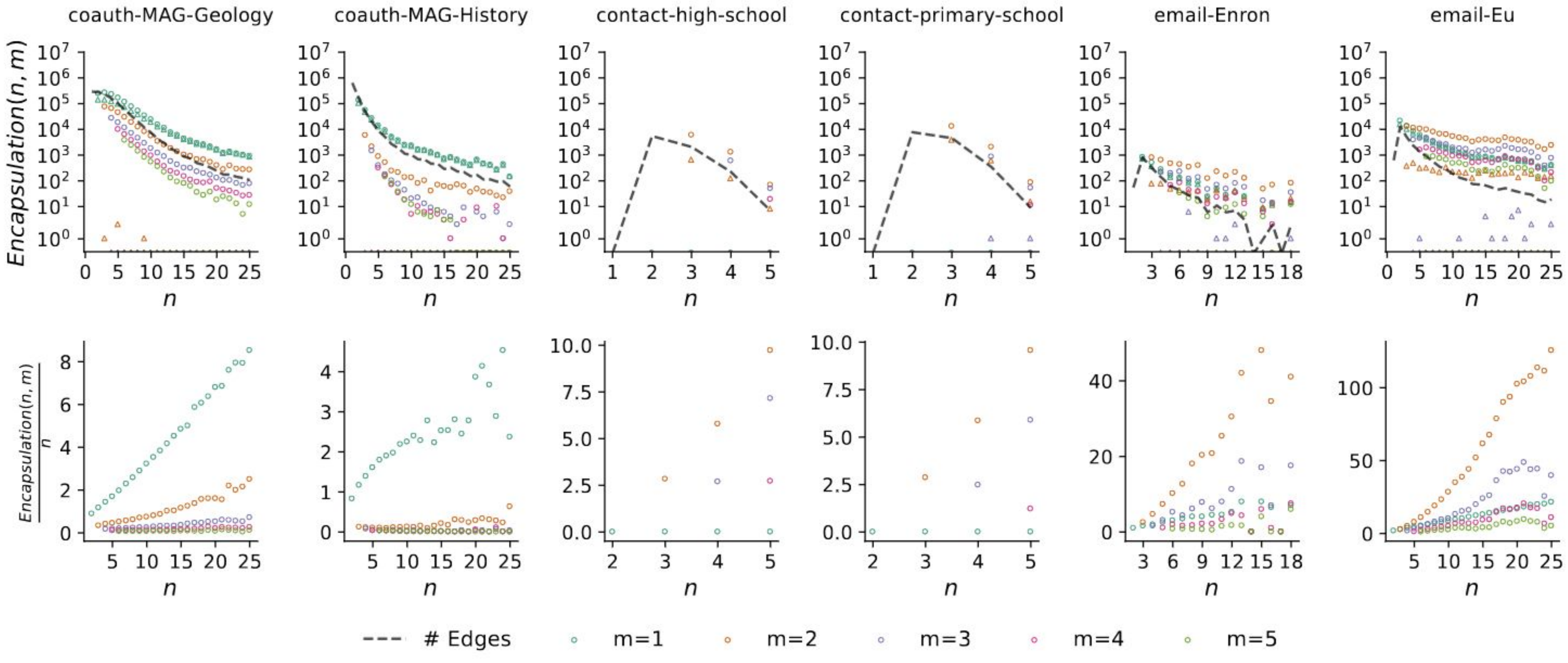
Timothy LaRock, Renaud Lambiotte

Future Work

- Structure
  - Temporal + dynamic interactions
  - Integration with work on hypergraph motifs, random walks, etc.
  - Further network analysis of encapsulation DAGs and other line graphs
- Dynamics
  - More complex models, including stochastic models
  - Combining node and hyperedge influence
  - Different models of influence between hyperedges (simulations w/ different line graphs)
  - Analytic results - encapsulation dynamics are node-based dynamics on line graphs
- Applications
  - Cooperation and competition
  - Social influence

Simulation Results

DAG Out-degree



Layer Randomization

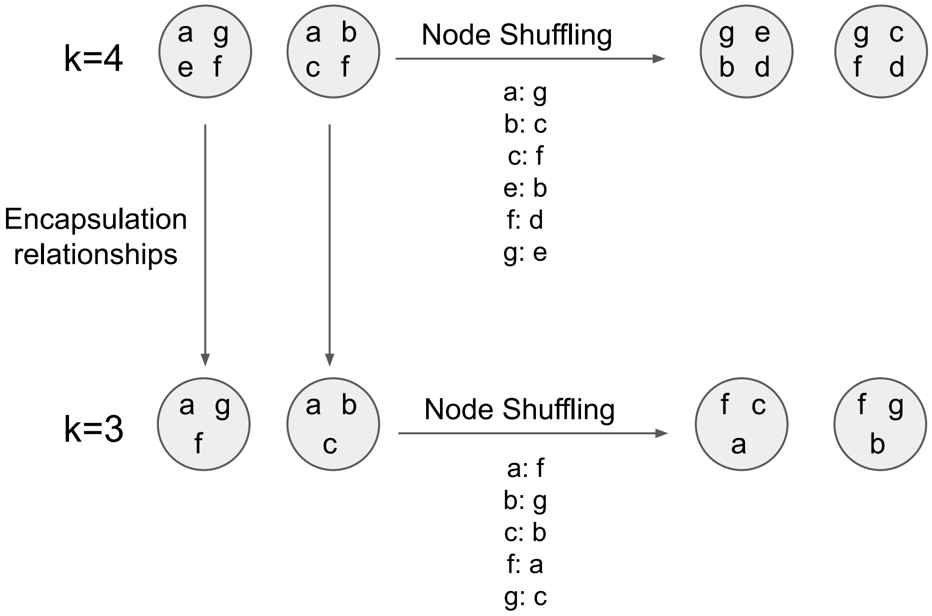
Idea: Shuffle node labels within each size layer

Randomizes:

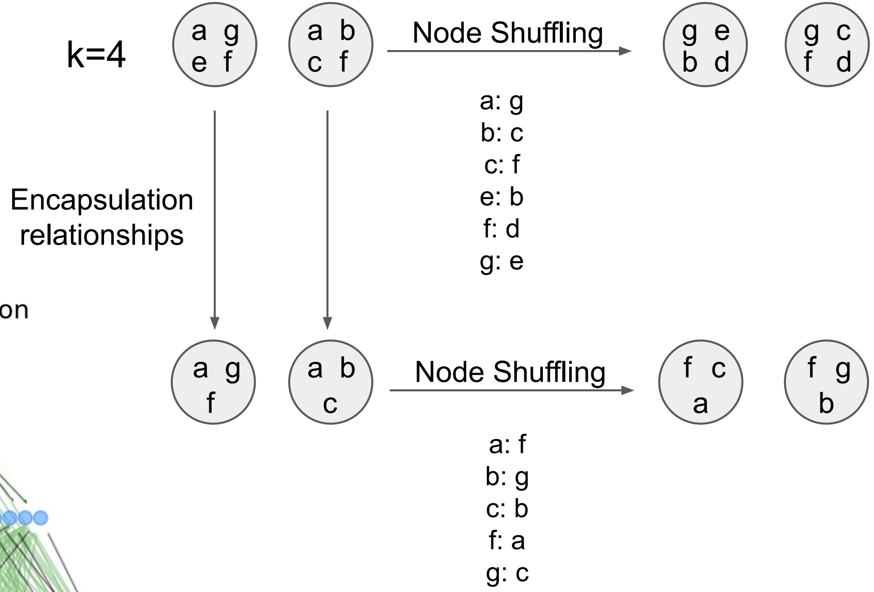
- Encapsulation and overlap relationships
- Labeled node-degree distributions within and across size layers

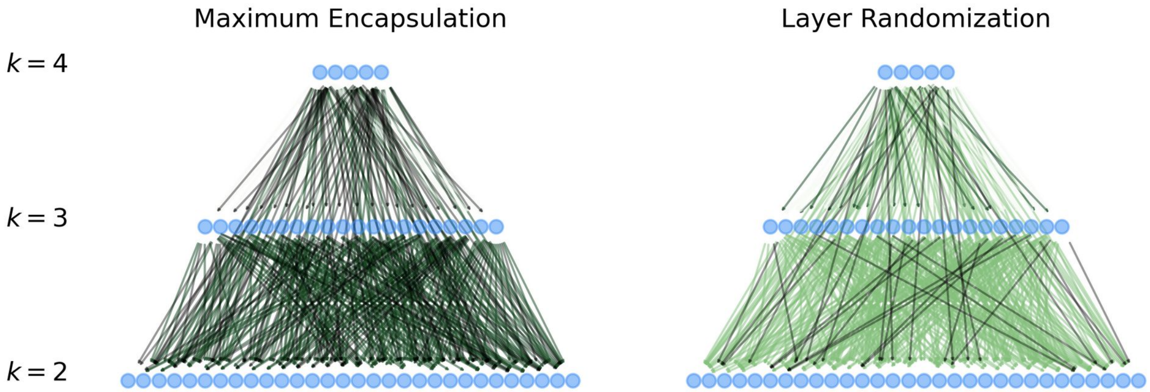
Preserves:

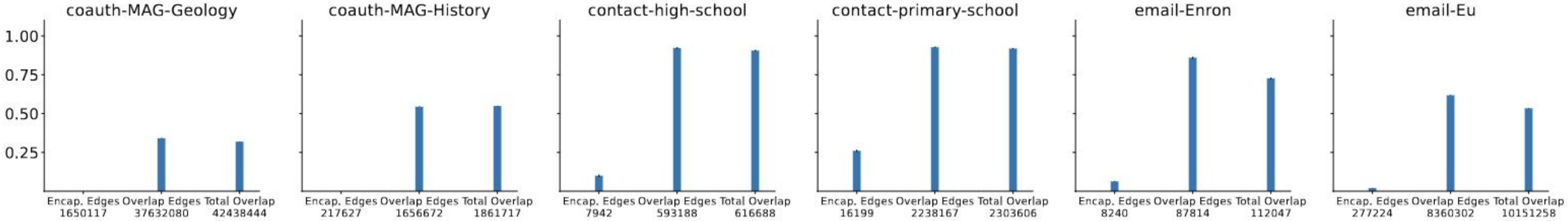
- Hyperedge size distribution
- Unlabeled node-degree distribution within size layers



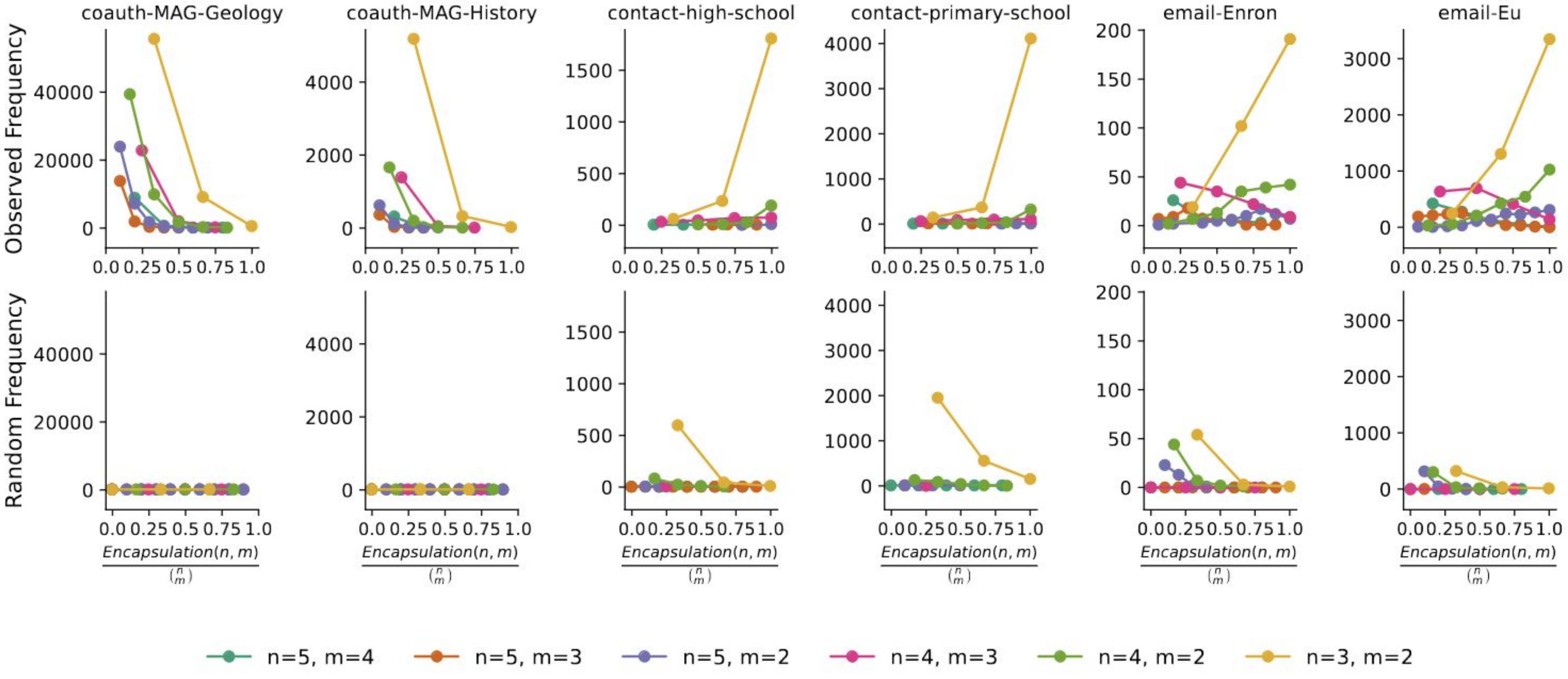
Layer Randomization



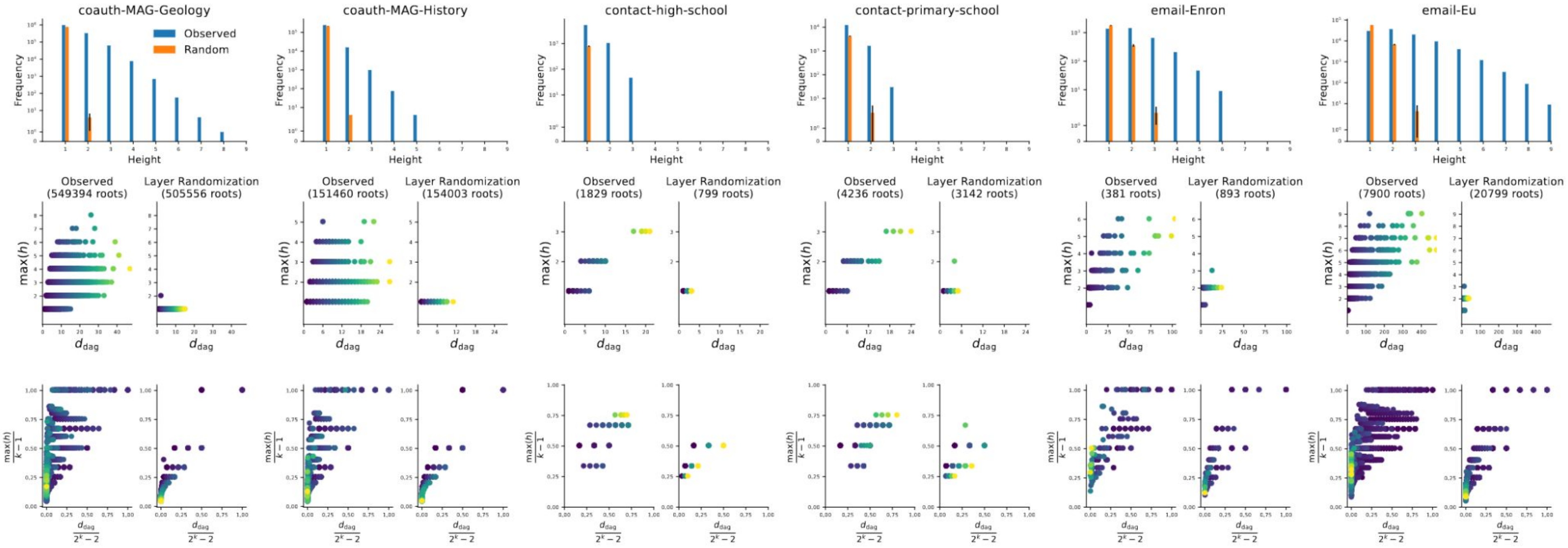




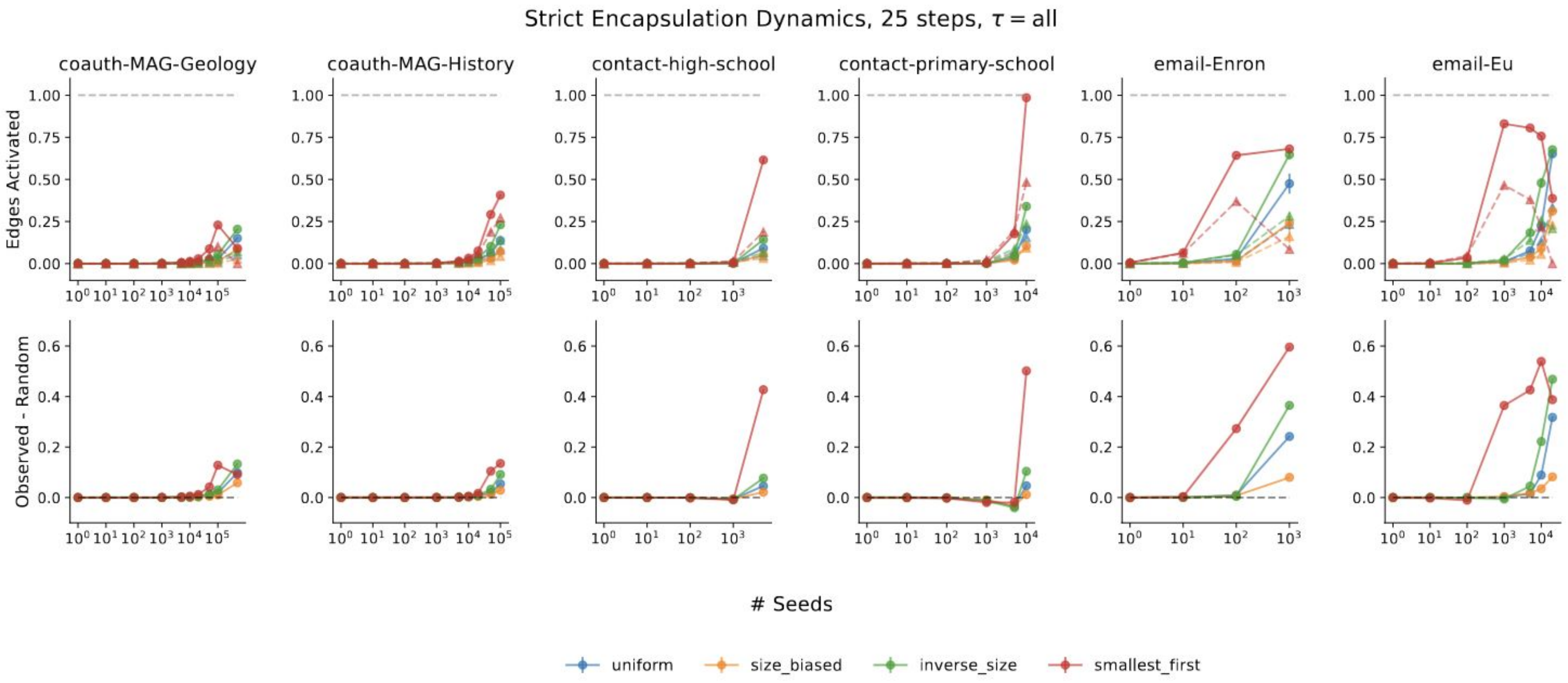
DAG Degree


coauth-MAG-Geology
coauth-MAG-History
contact-high-school
contact-primary-school
email-Enron
email-Eu
Observed Frequency
Random Frequency
0.0 0.25 0.5 0.75 1.0
Encapsulation(n, m)
$\binom{n}{m}$
n=5, m=4
n=5, m=3
n=5, m=2
n=4, m=3
n=4, m=2
n=3, m=2

Paths through DAGs



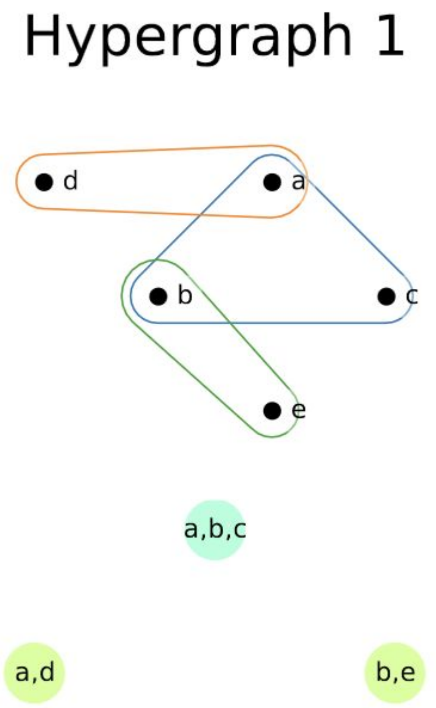
Empirical Simulation Results

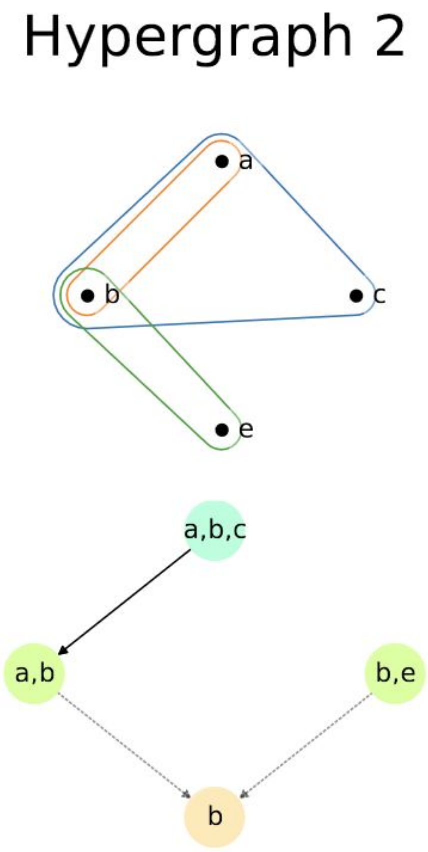


Encapsulation Dynamics

In node-based threshold dynamics, all hyperedges will become active if nodes a and b become activated.

Full activation will be impossible in encapsulation dynamics.





In node-based threshold dynamics, activating any of a,b,e could activate all hyperedges.

In encapsulation dynamics, full activation can only happen if hyperedge {a,b} is activated.

In non-strict dynamics, activation of b would activate a,b

There is tons of great work on higher-order structures and dynamics!

Networks beyond pairwise interactions: Structure and dynamics

Federico Battiston [a,*], Giulia Cencetti [b], Iacopo Iacopini [c,d], Vito Latora [c,e,f,g], Maxime Lucas [h,i,j], Alice Patania [k], Jean-Gabriel Young [l], Giovanni Petri [m,n]

Simplicial closure and higher-order link prediction

Austin R. Benson[a], Rediet Abebe[a], Michael T. Schaub[b,c], Ali Jadbabaie[b,d], and Jon Kleinberg[a,1]

Higher-order percolation processes on multiplex hypergraphs

Hanlin Sun[1] and Ginestra Bianconi[1,2]

How Do Hyperedges Overlap in Real-World Hypergraphs? - Patterns, Measures, and Generators

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Contagion dynamics on hypergraphs with nested hyperedges

Jihye Kim,[1] Deok-Sun Lee,[2,*] and Kwang-Il Goh[1,†]

[1] *Department of Physics, Korea University, Seoul 02841, Korea*
[2] *School of Computational Sciences and Center for AI and Natural Sciences, Korea Institute for Advanced Study, Seoul 02455, Korea*

Evolutionary dynamics of higher-order interactions in social networks

Unai Alvarez-Rodriguez[1,2,*], Federico Battiston[3,4], Guilherme Ferraz de Arruda[5], Yamir Moreno[5,6,7], Matjaž Perc[8,9,10], and Vito Latora[2,11,12]

Article
https://doi.org/10.1038/s41467-023-37118-3

Multistability, intermittency, and hybrid transitions in social contagion models on hypergraphs

Received: 13 April 2022
Accepted: 28 February 2023

Guilherme Ferraz de Arruda [1], Giovanni Petri [1,2], Pablo Martin Rodriguez [3] & Yamir Moreno [1,4,5]

High-order Line Graphs of Non-uniform Hypergraphs: Algorithms, Applications, and Experimental Analysis

Xu T. Liu*†, Jesun Firoz‡, Sinan Aksoy‡, Ilya Amburg‡, Andrew Lumsdaine†‡, Cliff Joslyn‡, Assefaw H. Gebremedhin†, Brenda Praggastis‡
*University of Washington, †Washington State University, ‡Pacific Northwest National Lab, USA
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There is tons of gre… on higher-order struc…res a… …mics!