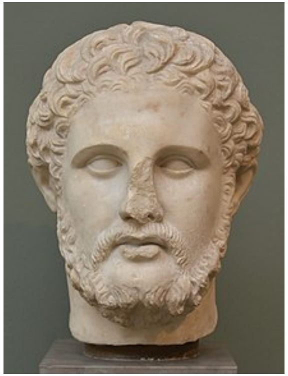
Lecture 2: Divide And Conquer

Section 1
Instructor Tim LaRock
larock.t@northeastern.edu
bit.ly/cs3000sylabus

Some business

No complaints about watching lectures via Canvas, going to keep doing it this way for now.

- Have fixed the layout so only the screen should be recorded
- Sharing screen directly from my iPad now, should go more smoothly (fingers crossed!)

Homework 1 to be released this evening; we will talk a bit about it at the end.

Decided against Discord/Slack, but also realized Canvas “discussions” are not full featured

- I will set up a Piazza instead (very sorry to do this late and add another thing!)

Student → TA assignment to come

Today

Some common growth functions, plotted

Loop invariants, take 2

We break things off with BubbleSort (feat. bad memes)

Introduction to Divide and Conquer

Very brief LaTeX “demo”

From last time: Asymptotes and Runtimes

"…an **asymptote** (/ˈæsɪmptoʊt/) of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the *x* or *y* coordinates tends to infinity." – Asymptote on Wikipedia

What do asymptotes have to do with algorithms?

From last time: Asymptotes and Runtimes

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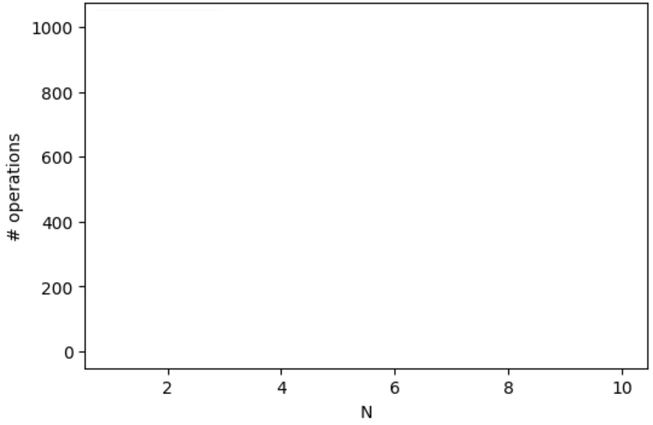

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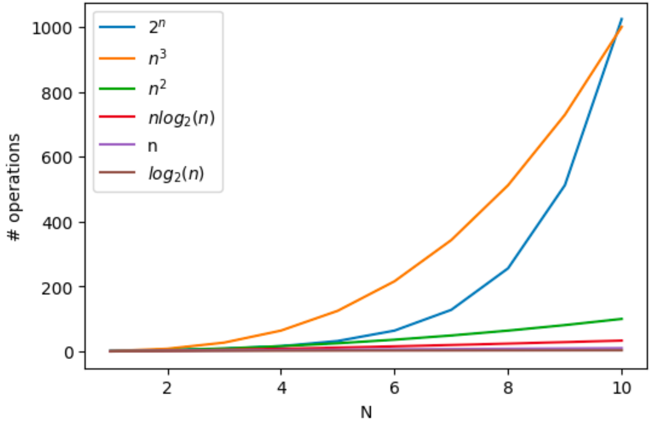

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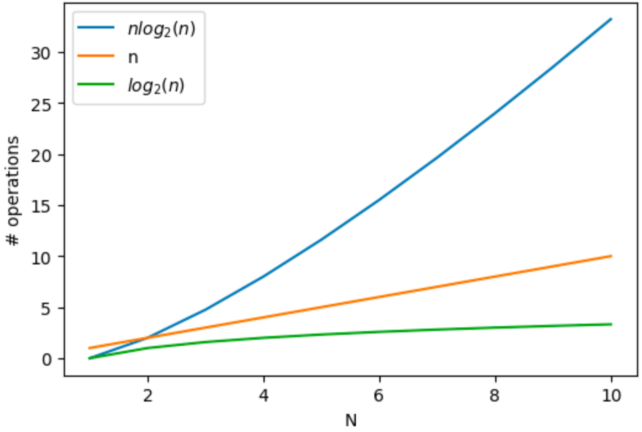

$O(n)$

From last time: Sorting

Sorting is extremely important to computer users and scientists!

A simple example: Finding the median of a set of numbers

```
Input: L, a list of N numbers
Output: The median of L
Procedure:
    1. Sort L
    2. If N is odd, return the number at L[⌈N/2⌉]
    3. If N is even, return the mean of the
       numbers at L[⌈N/2⌉] and L[⌈N/2⌉+1]
```

From last time: Bubble Sort

Idea: Items "bubble up" to the top as they are sorted pairwise

```
Input: L, a list of N numbers
Output: L sorted in ascending order
Procedure:
    Let swapped = True
    while swapped = True:
        swapped = False
        for i from 1 to N-1:
            if L[i] > L[i+1]:
                Swap L[i] and L[i+1]
                swapped = True
```

Best case time: $O(n)$

Worst case time: $O(n^2)$

Loop Invariant Definition

A loop invariant is a formal statement about the relationship between variables in [an algorithm] which holds true just before the loop is ever run (**establishing the invariant**) and is true again at the bottom of the loop, each time through the loop (**maintaining the invariant**).

Definition source: https://www.cs.miami.edu/home/burt/learning/Math120.1/Notes/LoopInvar.html

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Bubble sort loop invariant: After every iteration, the largest previously unsorted value is in its correct position.

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After *m* iterations of the while loop, the *m* largest values are in their correct positions.

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Bubble sort loop invariant: After every iteration, the largest previously unsorted value is in its correct position.

After *m* iterations of the while loop, the *m* largest values are in their correct positions.

After *n* iterations, all values are in their correct positions.

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Dumping BubbleSort: $O(n^2)$ is just not practical!

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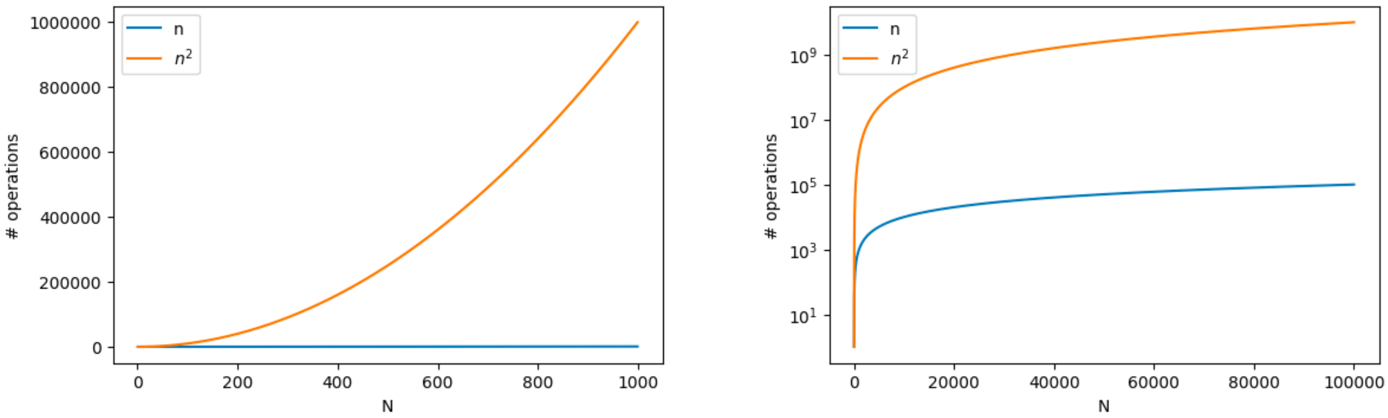


Dumping BubbleSort: O(n^2) is just not practical!

Enter: Divide and Conquer

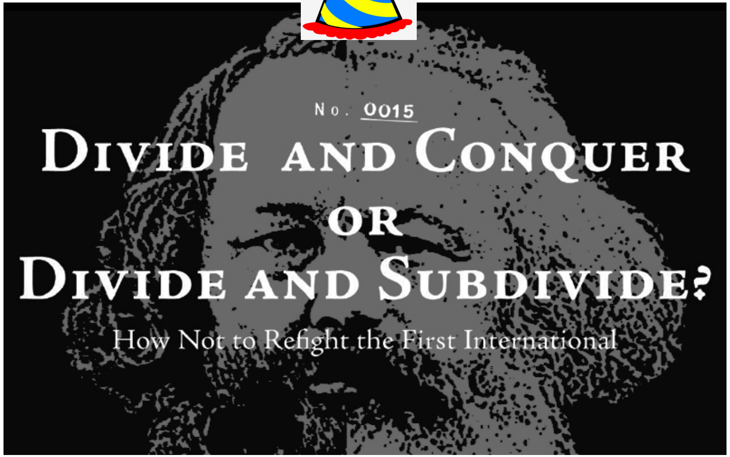
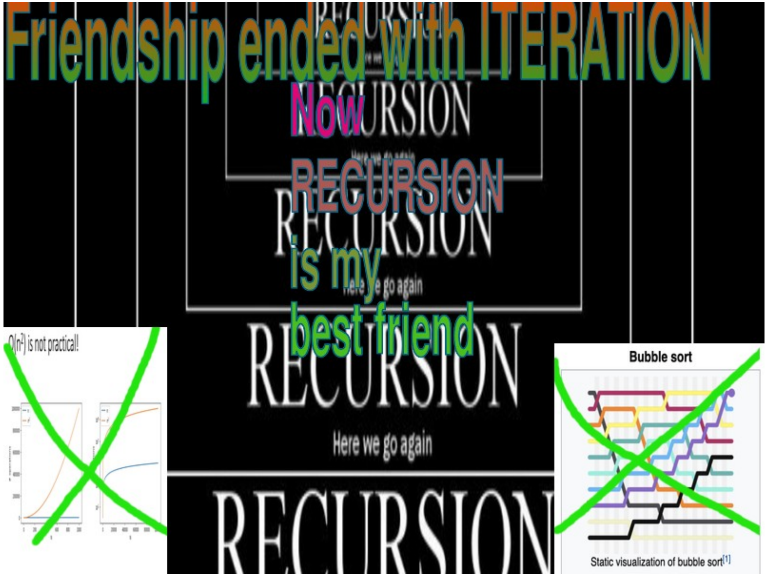


Image credit: libcom.org

What if….

Instead of sorting the entire input at once (as in bubble sort)….



…we could break the problem into smaller pieces to be sorted separately?

Merge Sort

Idea: Speed up sorting by splitting the input in half, sorting the smaller pieces separately, then merging the output.

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```
MERGESORT(A[1..n]):
    if n > 1
        m ← ⌊n/2⌋
        MERGESORT(A[1..m])          ⟨⟨Recurse!⟩⟩
        MERGESORT(A[m+1..n])        ⟨⟨Recurse!⟩⟩
        MERGE(A[1..n], m)
```

```
MERGE(A[1..n], m):
    i ← 1;  j ← m+1
    for k ← 1 to n
        if j > n
            B[k] ← A[i];  i ← i+1
        else if i > m
            B[k] ← A[j];  j ← j+1
        else if A[i] < A[j]
            B[k] ← A[i];  i ← i+1
        else
            B[k] ← A[j];  j ← j+1

    for k ← 1 to n
        A[k] ← B[k]
```

Erickson book section 1.4

Merge Sort Example

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5	3	6	2	2

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  for k ← 1 to n
      A[k] ← B[k]
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5 3 | 6 2 2

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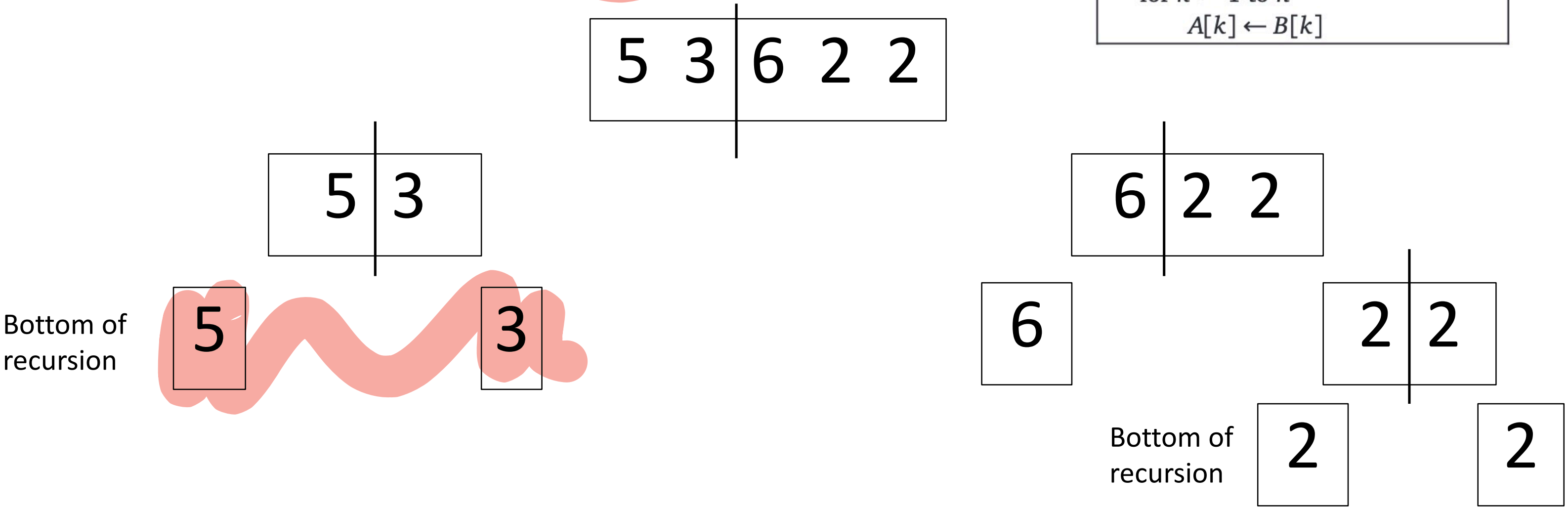

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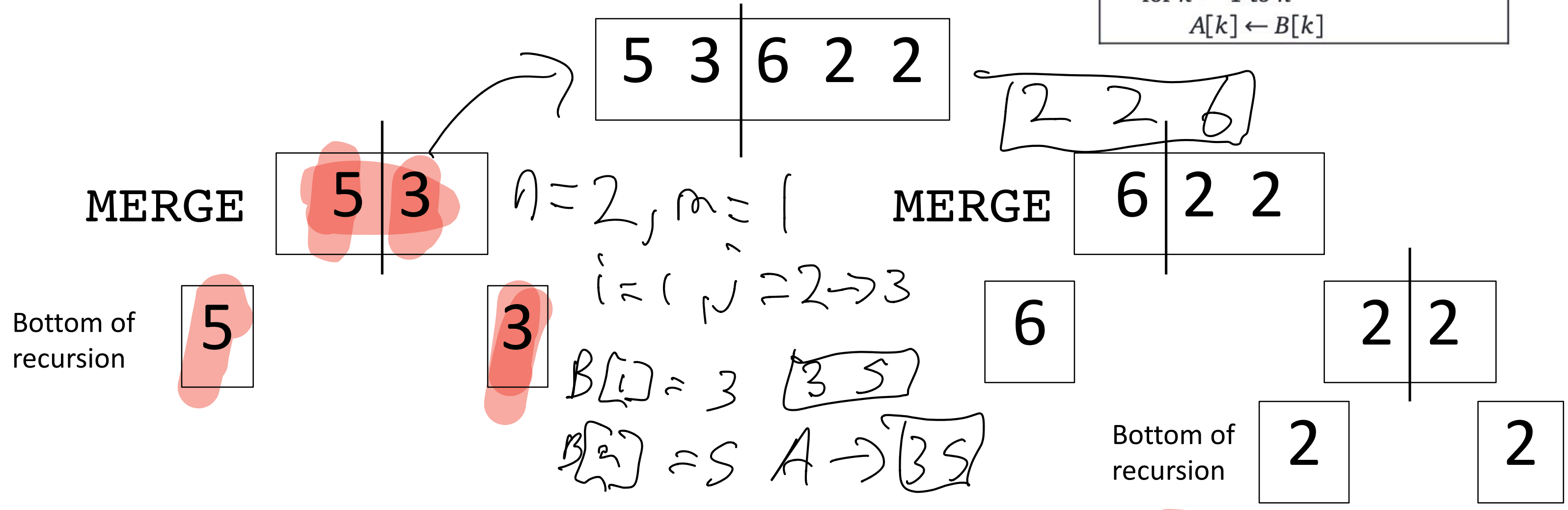

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226 | 35

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  for k ← 1 to n
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```

5	3	6	2	2

2 2 3 5 6

MERGE

3	5	2	2	6

i=1, j=3

i=2, j=3, j=4, j=5

i=3

B[1] = 2

B[2] = 2

B[3] = 3

B[4] = 5

B[5] = 6

Proof of Correctness

We can show formally that the output of MergeSort is correct by using 2 *proofs by induction*!

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Erickson book section 1.4

Proof by Induction Reminder

3 main steps to a proof by induction:

1) Show claim is true for base case.

inductive hypothesis

2) Assume claim is true for all $K<n$

3) Use inductive hypothesis to show claim is true for all K

Merge Sort: Proof of Correctness

First show that MERGE is correct, then MergeSort.

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  for k ← 1 to n
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```

Merge: Proof of Correctness

$m = \frac{n}{2} = 0$ $\quad n = 1 \quad j = 1 \quad j = m$

```
MERGE(A[1..n], m):
  i ← 1;  j ← m + 1
  for k ← 1 to n
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  for k ← 1 to n
      A[k] ← B[k]
```

We will show that for all k from 0 to n, the last n-k-1 iterations of the main loop correctly merge A[i..n] and A[j..m] into B[k..n].

Base case: $n = 1$

It is trivially true that Merge() on an array w/ 1 or 0 elements is correct.

Merge: Proof of Correctness

```
MERGE(A[1..n], m):
  i ← 1; j ← m+1
  for k ← 1 to n
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We will show that for all k from 0 to n, the last n-k-1 iterations of the main loop correctly merge A[i..n] and A[j..m] into B[k..n].

Inductive Hypothesis:

For an arbitrary iteration of the algorithm k, B[1,...,k-1] is correctly sorted.

Merge: Proof of Correctness

A[1..m] & A[m+1..n]

We will show that for all k from 0 to n, the last n-k-1 iterations of the main loop correctly merge A[i..n] and A[j..m] into B[k..n].

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MERGE(A[1..n], m):
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      else
          B[k] ← A[j]; j ← j + 1
  for k ← 1 to n
      A[k] ← B[k]
```

Consider an arbitrary iteration k+1

B[k+1]

Proof: 1) j > n means that the right subarray is exhausted of elements, so we put the next element from the left

MergeSort: Proof of Correctness

m relate to k+1

```
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  if n > 1
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    MERGESORT(A[m+1..n])      ⟨⟨Recurse!⟩⟩
    MERGE(A[1..n], m)
```

$m < k+1$

$\leq k$

Base Case: $n = 1$

Inductive Hypothesis:

For $1 \leq k < n$ MergeSort returns a correctly sorted array.

Proof:

Assume inductive hypothesis.

For $A[1 \ldots k+1]$ show MergeSort is correct.

$n \to \infty$

MergeSort: Runtime Analysis

$O(n \log_2 n)$

```
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  for k ← 1 to n
    A[k] ← B[k]
```

Let's write down a *recurrence relation* that describes the runtime:

$T(n) = T(\text{Division}) + T(\text{Conquer})$

$T(n) = O(2n) + T(\lfloor \frac{n}{2} \rfloor) + T(\lfloor \frac{n}{2} \rfloor + 1)$

$T(n) = 2 \cdot T(\frac{n}{2}) + O(n)$

Next Time

Recurrence Relations + Recurrence Trees

Formal Asymptotic Analysis

More Divide & Conquer

Suggested Readings:

Now: Brief LaTeX “demo”