Lecture 5: More Recursion

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bit.ly/cs3000syllabus

Business

- Homework 1 due tonight!
  - Only turn in compiled PDF, no need for .tex file
  - Make sure you turn something in!
    - It is okay, even expected, if you aren't totally sure about some solutions. Do your best.
    - Remember that 1 homework grade is dropped
- Homework 2 will be released tomorrow and due next Monday at midnight Boston time
- Office hours: Please email ahead of time with topic!
- Reminder: Use Piazza for questions as much as possible
  - You can ask private questions to the instructors. This is preferable to email.

Business 2: Exams

Tentative Schedule for Exams:

Midterm 1: Release next **Weds 5/20 8pm** and due **Friday 5/22 8pm**

Midterm 2 (tentative): Same deal starting Wed June 4

Final Exam TBD (probably either June 17-19 or during finals week)

Today

More recursion examples

- Selection without sorting
- Binary Search
- Master Theorem for solving recurrence relations

Finding the median without sorting

We motivated sorting with the median problem

```
Input: L, an array of N numbers
Output: The median of L
Procedure:
    1. Sort L
    2. If N is odd, return the number at L[⌈N/2⌉]
    3. If N is even, return the mean of the
       numbers at L[⌈N/2⌉] and L[⌈N/2⌉+1]
```

$O(n \log n)$

Can we compute the median without sorting the whole list first?

Selection without sorting

More general goal: Given unsorted array of integers `A`, how long to find the:

- Smallest number? $O(n)$
- Second smallest number? $O(2n) \rightarrow O(n)$
- k^{th} smallest number? $O(kn)$
- Median? $\left\lceil \frac{n}{2} \right\rceil$

A					
11	3	5	6	8	2

smallest
2nd smallest
3rd smallest
⋮
$\frac{n}{2}$nd smallest

$O(n \frac{n}{2}) \sim O(n^2)$

$O(n \log n)$

Selection without sorting

More general goal: Given unsorted array of integers `A`, how long to find the:

- Smallest number?
- Second smallest number?
- k^{th} smallest number?
- Median?

`A`					
11	3	5	6	8	2

Idea: What if we break the input array into subarrays in a "smart" way so that only 1 subarray needs to be searched recursively?

Today: Smart recursion for an $O(n)$ selection algorithm.

QuickSelect: Selection without sorting

Idea: Break the input array into subarrays in a "smart" way so that only 1 subarray needs to be searched recursively

```
QuickSelect(A[1..n], k):
  If n = 1:
    return A[1]
  Else:
    Choose a pivot element A[p]
    r ← Partition(A, p)

    If k < r:
      Return QuickSelect(A[1..r], k)
    ElseIf k > r:
      Return QuickSelect(A[r+1..n], k-r)
    Else:
      Return A[r]
```

minimum, $k=1$

maximum, $k=n$

median, $k=\frac{n}{2}$

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      Return A[r]
```

→ Given A and p, return the array transformed so that all elements in the left half are less than A[p], the middle value is A[p] , and all the elements in the right half are greater than A[p]

A	11	3	5	4	8	2

QuickSelect: Selection without sorting

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```
QuickSelect(A[1..n], k):
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    Else:
      Return A[r]
```

$p = 3$

Given A and p, return the array transformed so that all elements in the left half are less than A[p], the middle value is A[p] , and all the elements in the right half are greater than A[p]

A	11	3	**5**	4	8	2

Pivot Element

QuickSelect: Selection without sorting

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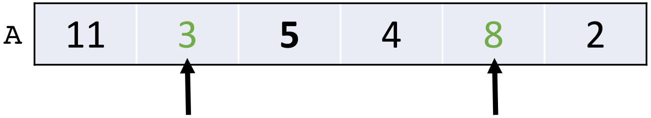
Given A and p, return the array transformed so that all elements in the left half are less than A[p], the middle value is A[p] , and all the elements in the right half are greater than A[p]



Already on the correct side of A[p]

QuickSelect: Selection without sorting

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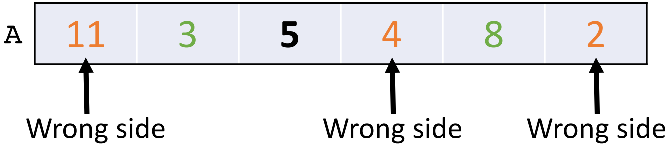

QuickSelect: Selection without sorting

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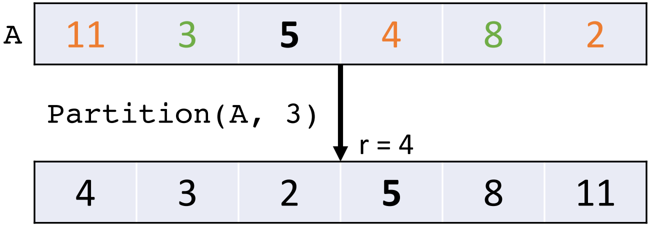

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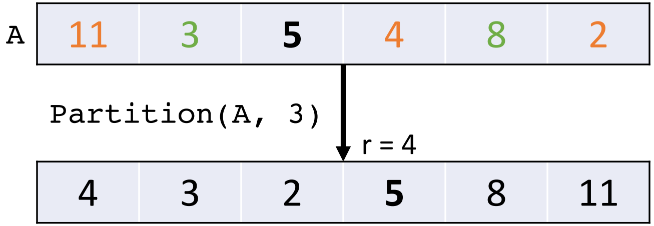

Note: Partitioning does **not** sort the array!

QuickSelect: Selection without sorting

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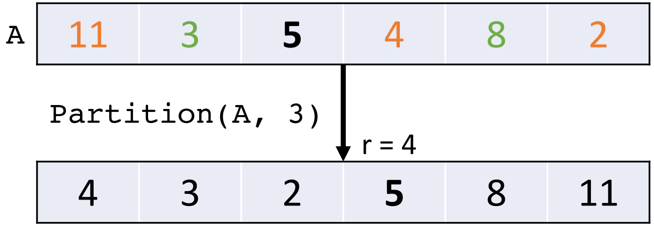
```

Key Observation: If I want the 3rd smallest value in this example (4) this partitioning scheme guarantees it is in the left subarray!

Given A and p, return the array transformed so that all elements in the left half are less than A[p], the middle value is A[p] , and all the elements in the right half are greater than A[p]



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QuickSelect Example

Assume `Partition(A, p)` is correct and I will "randomly" choose pivots

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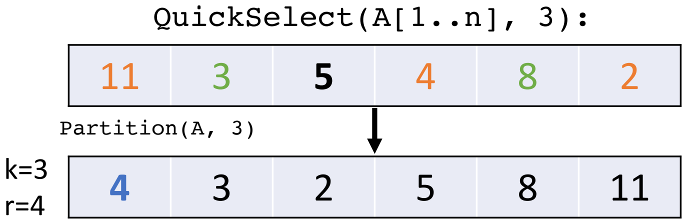

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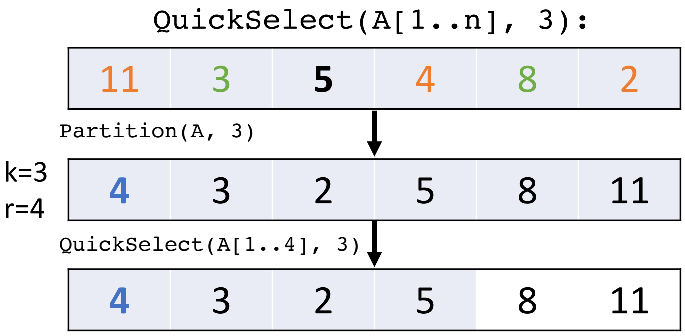

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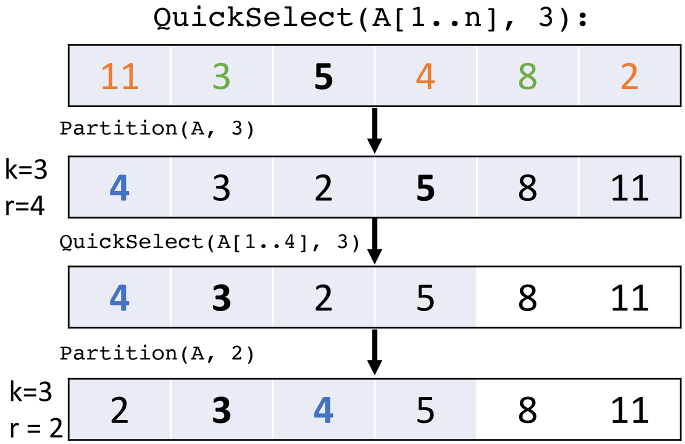

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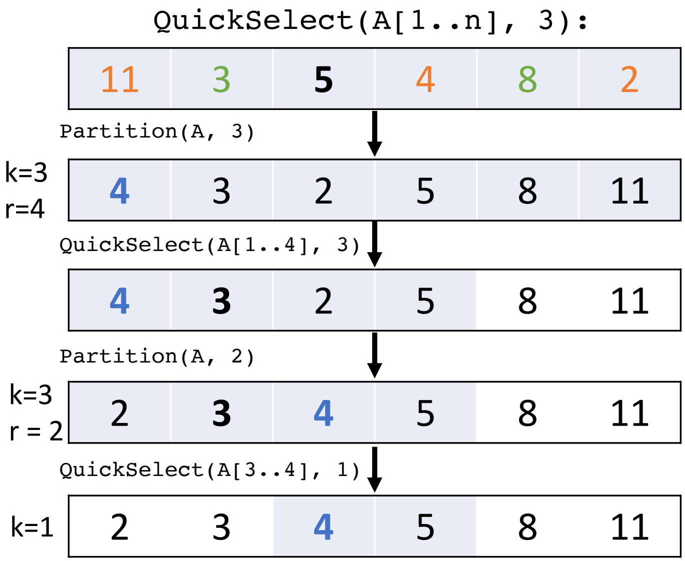

Two possibilities:

QuickSelect Example

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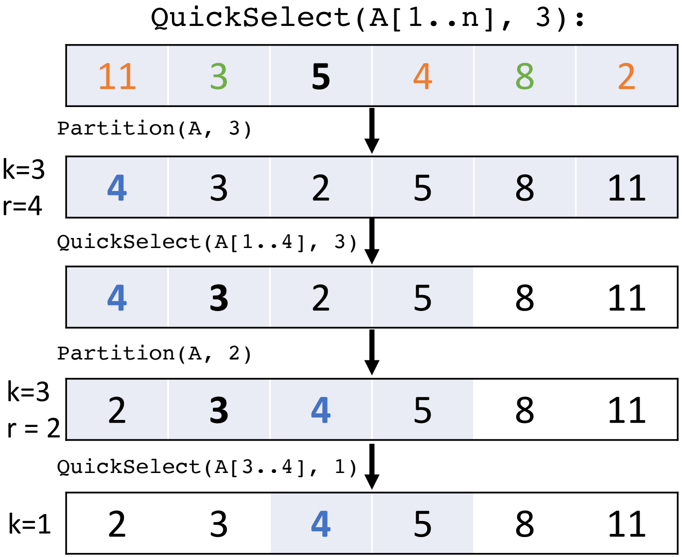

Two possibilities:

1. We pivot on 4 (r=1), in which case r=k and we return A[1] = 4

QuickSelect Example

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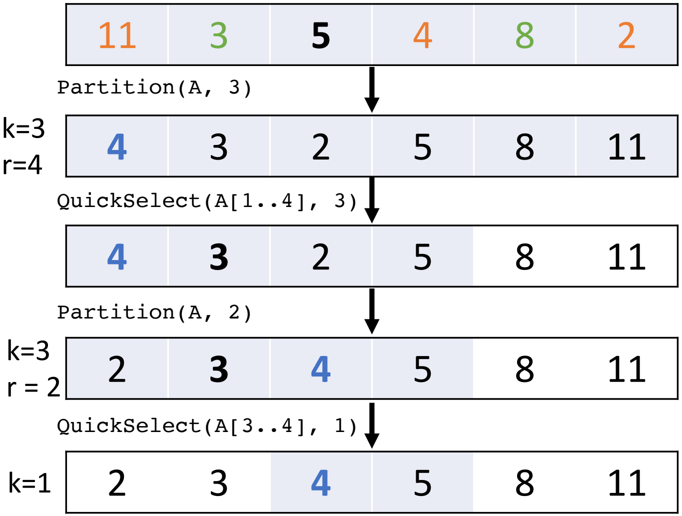
```

looking for the k-th smallest element

QuickSelect(A[1..n], 3):



Two possibilities:

1. We pivot on 4 (r=1), in which case r=k and we return A[1] = 4
2. We pivot on 5 (r=2), in which case we recurse on just 4, meaning n=1 and we return 4

QuickSelect: Choosing pivot elements

Problem: How do we choose a "good" pivot element?

```
QuickSelect(A[1..n], k):
  If n = 1:
    return A[1]
  Else:
    Choose a pivot element A[p]
    r ← Partition(A, p)  → O(n)

    If k < r:
      Return QuickSelect(A[1..r], k)
    ElseIf k > r:
      Return QuickSelect(A[r+1..n], k-r)
    Else:
      Return A[r]
```

1 2 3 4 5

- What happens if you choose the minimum value as the pivot? Or maximum value?
- Without assuming anything about the input array, it is difficult to pick a good pivot *a priori*!
- What is our goal for a "good pivot"?

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```

- What happens if you choose the minimum value as the pivot? Or maximum value?
- Without assuming anything about the input array, it is difficult to pick a good pivot *a priori*!
- What is our goal for a "good pivot"?
  - Close to the median!

Median of Medians

Idea: Choose a pivot element by approximating the median.

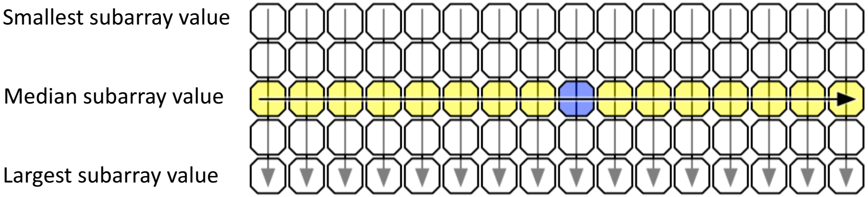
```
MOM(A[1..n]):
  Let m ← ⌈n/5⌉
  For i in 1,..,m:
    Medians[i] = Median(A[5i-4..5i])
  med ← MOMSelect(Medians[i..m], ⌊m/2⌋)
  return index of med in A
```

Break the input up into $\left\lceil \frac{n}{5} \right\rceil$ subarrays, take the median of each, then find the median of those medians (MoM).

Median of Medians

- **Claim:** For every $\boldsymbol{A}$ there are at least $3n/10$ items that are smaller than $\mathbf{MOM}(\boldsymbol{A})$ and at least $3n/10$ items that are larger.

n=80



Visualizing the median of medians

Median of Medians

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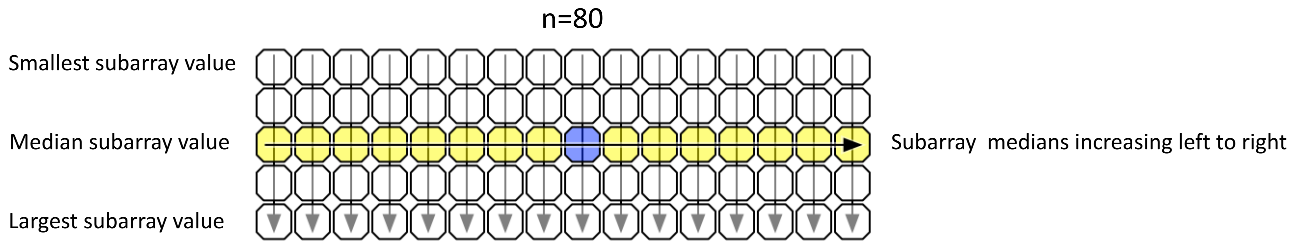

Visualizing the median of medians

Median of Medians

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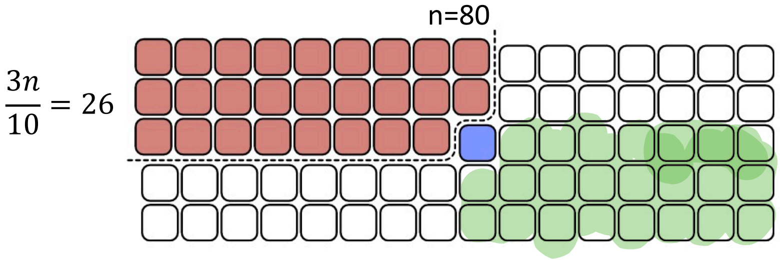
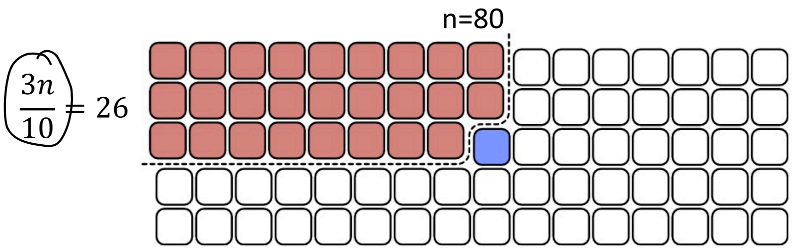


Median of Medians

- **Claim:** For every $\boldsymbol{A}$ there are at least $3n/10$ items that are smaller than $\mathbf{MOM}(\boldsymbol{A})$ and at least $3n/10$ items that are larger.

$m = \frac{n}{5} = \frac{80}{5} = 16$



$n - \frac{3n}{10} = \frac{10n}{10} - \frac{3n}{10} = \frac{7n}{10}$

- If k is smaller than $\frac{3n}{10}$, recurse on those items
- If k is larger than $\frac{3n}{10}$, recurse on the remaining $n - \frac{3n}{10} = \frac{7n}{10}$ items

MOMSelect

```
MOMSelect(A[1..n], k):
  If n <= 25:
    return median(A)
  Else:
    mom ← MOM(A[1..n])
    r ← Partition(A, mom)

    If k < r:
      Return MOMSelect(A[1..r], k)
    ElseIf k > r:
      Return MOMSelect(A[r+1..n], k-r)
    Else:
      Return A[r]
```

→ within this there is a MOMSelect($\frac{n}{5}$)

MOMSelect Running Time

```
MOMSelect(A[1..n], k):
  If n <= 25:
    return median(A)
  Else:
    mom ← MOM(A[1..n])
    r ← Partition(A, mom)   --> O(n)

    If k < r:
      Return MOMSelect(A[1..r], k)
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      Return MOMSelect(A[r+1..n], k-r)
    Else:
      Return A[r]
```

What is a recurrence relation for MOMSelect?

T(n) = T(Selection) + T(MOM) + f(ops per step)

$$T(n) = T\left(\frac{7N}{10}\right) + T\left(\frac{n}{5}\right) + O(n)$$

$$= T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n)$$

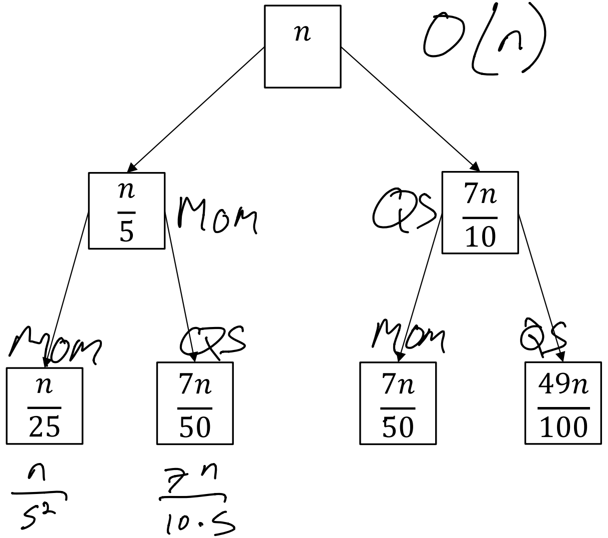
Recursion Tree

$$T(n) = T(\tfrac{7n}{10}) + T(\tfrac{n}{5}) + O(n)$$

$f(n)$

$$\sum_{\ell=0}^{L} \left(\frac{1}{5}\right)^{\ell} n + \left(\frac{7}{10}\right)^{\ell} n$$

$O(n)$ $O(n)$



$$\left(\frac{9}{10}\right)n$$

$$\frac{81}{100} = \left(\frac{9}{10}\right)^2 n$$

$$\frac{2n}{10}$$

$$\frac{n}{5} + \frac{7n}{10} = \frac{9n}{10} = \frac{9}{10}n$$

$$\frac{4n + 14n + 14n + 49n}{100} = \left(\frac{81}{100}\right)n = \left(\frac{9}{10}\right)^2 n$$

Since the work at each level is decreasing exponentially, the O(n) term dominates!

Proof by induction

T(n) = T($\frac{7n}{10}$) + T($\frac{n}{5}$) + O(n)
T(1) = 1

We want to show that

$T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n) \leq O(n)$, meaning

Proof by induction

T(n) = T($\frac{7n}{10}$) + T($\frac{n}{5}$) + O(n)
T(1) = 1

Assume $T(n) \leq C \cdot n$ for all power values of n

We want to show that

$$T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n) \leq O(n), \text{ meaning}$$

$$T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + n \leq Cn \text{ (for some } C\text{)}$$

Proof by induction

T(n) = T($\frac{7n}{10}$) + T($\frac{n}{5}$) + O(n)
T(1) = 1

We want to show that

$$T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n) \leq O(n),$$ meaning

$$T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + n \leq Cn$$ (for some C)

By induction, since $\frac{1}{5}n < \frac{7}{10}n < n$, we have

$$C\frac{7n}{10} + C\frac{n}{5} + n$$

Proof by induction

T(n) = T($\frac{7n}{10}$) + T($\frac{n}{5}$) + O(n)
T(1) = 1

We want to show that

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By induction, since $\frac{1}{5}n < \frac{7}{10}n < n$, we have

$$C\frac{7n}{10} + C\frac{n}{5} + n$$

Pulling out n, we get

$$n\left(C\frac{7}{10} + C\frac{1}{5} + 1\right)$$

$$n\left(C\frac{9}{10} + 1\right)$$

$$\leq Cn$$

Proof by induction

T(n) = T($\frac{7n}{10}$) + T($\frac{n}{5}$) + O(n)
T(1) = 1

We want to show that

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$$n\left(C\frac{7}{10} + C\frac{1}{5} + 1\right)$$

$$n\left(C\frac{9}{10} + 1\right)$$

$$\leq Cn$$

For which values of C?

$$C\frac{9}{10} + 1 \leq C$$

$$9C + 10 \leq 10C$$

$$C \geq 10$$

Proof by induction

T(n) = T($\frac{7n}{10}$) + T($\frac{n}{5}$) + O(n)
T(1) = 1

We want to show that

$$T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n) \leq O(n), \text{ meaning}$$

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By induction, since $\frac{1}{5}n < \frac{7}{10}n < n$, we have

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Pulling out n, we get

$$n\left(C\frac{7}{10} + C\frac{1}{5} + 1\right)$$

$$n\left(C\frac{9}{10} + 1\right)$$

$$\leq Cn \quad \text{(as long as } C \geq 10\text{)}$$

For which values of C?

$$C\frac{9}{10} + 1 \leq C$$

$$9C + 10 \leq 10C$$

$$C \geq 10$$

MOMSelect Wrap

$O(n \log n)$

S is the minimum s.t. we get $O(n)$

- We can find the median of a list of numbers in $O(n)$ time (faster than sorting) using divide and conquer approach
- Key: Selecting a good pivot with median-of-medians-of-five
- This technique also works for sorting (QuickSort) in $O(nlogn)$

```
MOMSelect(A[1..n], k):
  If n <= 25:
    return median(A)
  Else:
    mom ← MOM(A[1..n])
    r← Partition(A, mom)

    If k < r:
      Return MOMSelect(A[1..r], k)
    ElseIf k > r:
      Return MOMSelect(A[r+1..n], k-r)
    Else:
      Return A[r]
```

Switching gears: Searching

Searching

Given a sorted array, what is the run time to find an element?

2	3	4	**5**	8	11

$O(n)$

Searching

Given a sorted array, what is the run time to find an element?

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Can we do it faster?

Binary Search

Idea: We can use the fact that the array is sorted to be smart about choosing the next subarray to search!

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  // A[1:n] sorted in ascending order
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Search(A,ℓ,r,t):
  If(ℓ > r): return FALSE

  m ← ℓ + ⌊(r−ℓ)/2⌋

  If(A[m] = t): return m
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```

`StartSearch(A,5):`

ℓ=1
r=6
m=3

2	3	4	**5**	8	11

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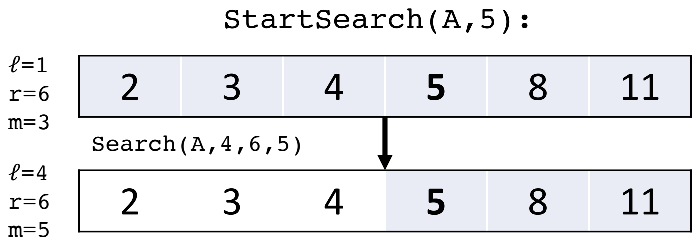

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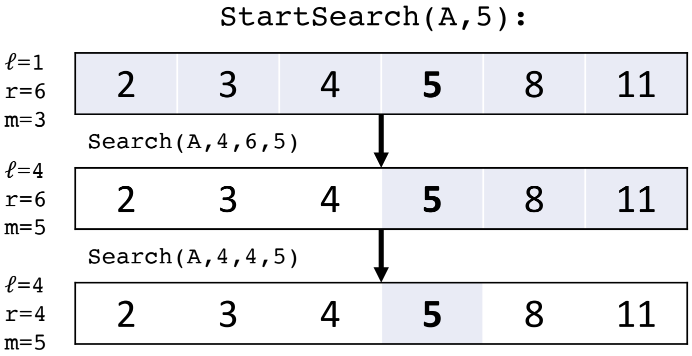

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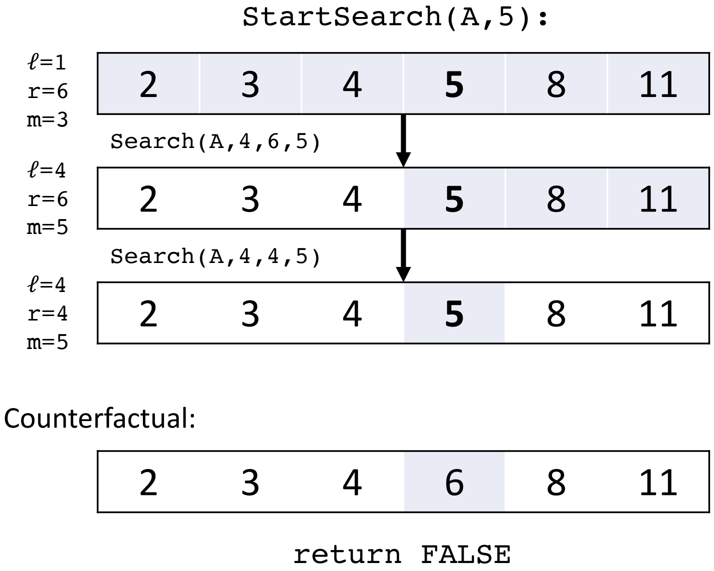

Binary Search Recurrence Relation

What does the recurrence relation look like for binary search?

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$T\left(\frac{n}{2}\right)$

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$T(n) = T(\frac{n}{2}) + O(1)$

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$T(n) = T(\frac{n}{2}) + O(1)$

We could use a recursion tree to get the running time, but there is also a more general result we can use…

Master Theorem

- Recipe for recurrences of the form:
  - $T(n) = \boldsymbol{a} \cdot T(n/\boldsymbol{b}) + Cn^{\boldsymbol{d}}$
- Three cases:
  - $\left(\frac{\boldsymbol{a}}{\boldsymbol{b}^{\boldsymbol{d}}}\right) > 1 : \; T(n) = \Theta\left(n^{\log_{\boldsymbol{b}} \boldsymbol{a}}\right)$
  - $\left(\frac{\boldsymbol{a}}{\boldsymbol{b}^{\boldsymbol{d}}}\right) = 1 : \; T(n) = \Theta\left(n^{\boldsymbol{d}} \log n\right)$
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T(n) = T($\frac{n}{2}$) +O(1)

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$T(n) = T(\frac{n}{2}) + O(1)$

$T(n) = 1T(\frac{n}{2}) + n^0$

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$$\left(\frac{1}{2^0}\right) = 1$$

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T(n) = T($\frac{n}{2}$) +O(1)

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So

$$T(n) = \Theta\left(n^{\mathbf{0}} \log n\right)$$

and we get

$$T(n) = \Theta(\log n)$$

Master Theorem

$O(n^d)$

there exists some C

s.t.

- Recipe for recurrences of the form:
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  - $\left(\frac{a}{b^d}\right) < 1: \ T(n) = \Theta\left(n^d\right)$

Note that the theorem does not apply to our MOMSelect recurrence:

$$\mathrm{T(n) = T(\tfrac{7n}{10}) + T(\tfrac{n}{5}) + O(n)}$$

Binary Search:

$T(n) \leq c \cdot n^d$

T(n) = T($\frac{n}{2}$) +O(1)

T(n) = 1T($\frac{n}{2}$) + n^0

$$\left(\frac{1}{2^0}\right) = 1$$

So

$$T(n) = \Theta\left(n^0 \log n\right)$$

and we get

$$T(n) = \Theta(\log n)$$

Wrap up

Homework 1 due tonight

Homework 2 will be released at 8AM

Next time:

- Backtracking
- Fibonacci numbers
- Dynamic Programming

Reading Assign

Erickson

Chapter 3

Ask the Audience!

- Use the Master Theorem to Solve:
  - $T(n) = 16 \cdot T\left(\frac{n}{4}\right) + n^2$
  - $T(n) = 21 \cdot T\left(\frac{n}{5}\right) + n^2$
  - $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + 1$
  - $T(n) = 1 \cdot T\left(\frac{n}{2}\right) + 1$