Lecture 18: Greedy Algorithms + Midterm Review

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bit.ly/cs3000syllabus

Business

Homework 5 due tonight at midnight Boston time, solutions will be released tomorrow morning

No class tomorrow, midterm review moved to today

Extra credit assignment available as of yesterday

- Optional
- 6 points on the final exam
- Available until Sunday June 21st

Midterm 2 to be released tomorrow night, due Friday night

- Topics: Graph algorithms and network flow

Greedy Algorithms

- For some problems, we can think of simple decision making rules that intuitively guide us towards a solution
  - Best-first search: We want to find shortest paths/minimum trees, so only choose edges that can be included in these solutions!
- Applying this idea does not always work as intended!
  - Maximum flow: We tried assigning flow based on best-first search, but we showed that the algorithm will get stuck if it is not able to modify the flow!
- Algorithms that rely on repeatedly making optimal *local decisions* to eventually reach an optimal global solution are called *greedy algorithms*

Example: Files on Tape

Before any of us were born, computers used to exist on magnetic tape.

Imagine we have such a tape, split in to segments we will call “blocks”, where each block contains data from a single file. Each file is referred to by an integer index i, and has length in blocks $L[i]$.

1	1	1	2	2	3	3	3	4	4

To read file k, the tape head needs to first skip all of the files before k. Therefore, the *cost* of accessing file k can be written as

$$cost(k) = \sum_{i=1}^{k} L[i]$$

Example: Files on Tape

1	1	1	2	2	3	3	3	4	4

Assuming all files are equally likely to be accessed, we can write the *expected* (equivalently, average) cost of accessing file *k* as

$$\mathbb{E}[cost] = \frac{1}{n}\sum_{i=1}^{n} cost(i) = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{k} L[i]$$

$$\mathbb{E}[cost] = \frac{1}{4} \cdot (cost(1) + cost(2) + cost(3) + cost(4))$$

Example: Files on Tape

$$= \frac{1}{4} \cdot (3 + 5 + 8 + 10) = \frac{26}{4}$$

1	1	1	2	2	3	3	3	4	4

Assuming all files are equally likely to be accessed, we can write the *expected* (equivalently, average) cost of accessing file *k* as

$$\mathbb{E}[cost] = \frac{1}{n}\sum_{i=1}^{n} cost(i) = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{k} L[i]$$

What order should we keep the files in?

1	1	1	2	2	3	3	3	4	4

$\mathbb{E}[cost] = \frac{26}{4}$

We can modify the order of the files on the tape, resulting in a permutation π where $\pi(i)$ returns the index of the file in the ith block. We can then rewrite the *expected* (average) cost of accessing file *k* as

$$\mathbb{E}[cost(\pi)] = \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{k} L[\pi(i)]$$

Intuitively: To minimize average cost, we should store the smallest files first, otherwise we will need to unnecessarily spend time skipping the large files to read smaller ones!

2	2	4	4	1	1	1	3	3	3

What order should we keep the files in?

1	1	1	2	2	3	3	3	4	4

$\mathbb{E}[cost] = \frac{26}{4}$

We can modify the order of the files on the tape, resulting in a permutation π where $\pi(i)$ returns the index of the file in the ith block. We can then rewrite the *expected* (average) cost of accessing file *k* as

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2	2	4	4	1	1	1	3	3	3

$$\mathbb{E}[cost(\pi)] = \frac{2+4+7+10}{4} = \frac{23}{4}$$

Greedy Algorithm for Storing Files

Input: A set of files labeled $1 \ldots n$ with lengths $L[i]$
Output: An ordering of the files on the tape

Repeat until all files are on the tape:

1. Find the unwritten file with minimum length (break ties arbitrarily)
2. Write that file to the tape

Greedy Algorithm for Storing Files

Input: A set of files labeled $1 \ldots n$ with lengths $L[i]$
Output: An ordering of the files on the tape

Repeat until all files are on the tape:

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How can we show this is optimal?

Proof of optimality

1	1	1	2	2	3	3	3	4	4

Claim: $\mathbb{E}[cost(\pi)]$ is minimized when $L[\pi(i)] \leq L[\pi(i+1)]$ for all i.

Proof of optimality

1	1	1	2	2	3	3	3	4	4

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Proof:

Let $\mathrm{a} = \pi(i)$ and $b = \pi(i+1)$ and suppose $L[a] > L[b]$ for some index i.

Proof of optimality

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Proof:

Let $\mathrm{a} = \pi(i)$ and $b = \pi(i+1)$ and suppose $L[a] > L[b]$ for some index i.

If we swap the files a and b on the tape, then the cost of accessing a increases by $L[b]$ and the cost of accessing b decreases by $L[a]$.

Proof of optimality

1	1	1	2	2	3	3	3	4	4

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If we swap the files a and b on the tape, then the cost of accessing a increases by $L[b]$ and the cost of accessing b decreases by $L[a]$.

Overall, the swap changes the expected cost by $\frac{L[b]-L[a]}{n}$.

Proof of optimality

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This change represents an improvement because $L[b] < L[a]$.

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This change represents an improvement because $L[b] < L[a]$.

Average cost for example above: $\frac{26}{4}$

Average cost after swapping files 1 and 2: $\frac{1}{4}(2+5+8+10) = \frac{25}{4}$

$$\frac{26}{4} + \frac{2-3}{4} = \frac{26-1}{4} = \frac{25}{4}$$

Proof of optimality

1	1	1	2	2	3	3	3	4	4

Claim: $\mathbb{E}[cost(\pi)]$ is minimized when $L[\pi(i)] \leq L[\pi(i+1)]$ for all i.

Proof:

Let $\mathrm{a} = \pi(i)$ and $b = \pi(i+1)$ and suppose $L[a] > L[b]$ for some index i.

If we swap the files a and b on the tape, then the cost of accessing a increases by $L[b]$ and the cost of accessing b decreases by $L[a]$.

Overall, the swap changes the expected cost by $\frac{L[b]-L[a]}{n}$.

This change represents an improvement because $L[b] < L[a]$.

Thus, if the files are out of length-order, we can decrease expected cost by swapping pairs to put them in order.

Wrap-up

Greedy algorithms repeatedly apply a simple rule to eventually find an optimal solution

Inductive Exchange Arguments are strategies for proving correctness of some greedy algorithms

Next Week:

- Data Compression with Huffman Codes
- Proof strategies for greedy algorithms
  - Inductive exchange
  - Greedy-stays-ahead

Midterm 2 Review/Q&A

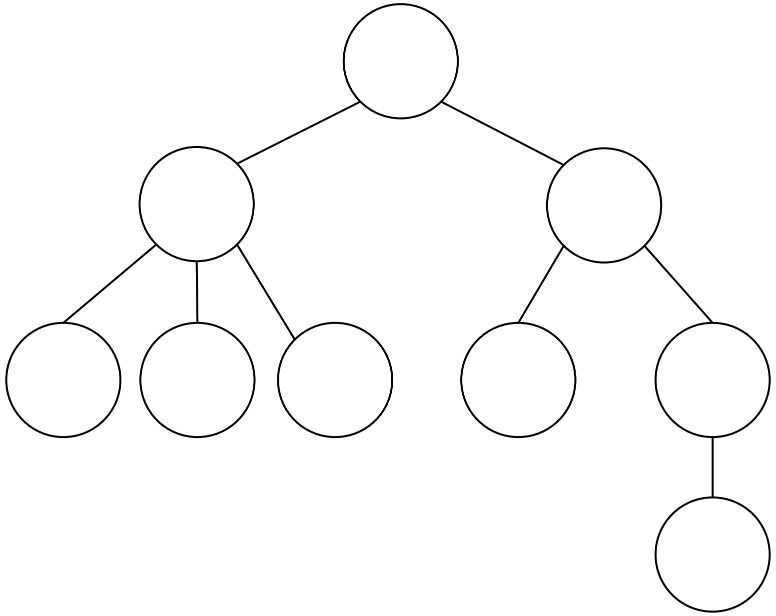
Topics

- Graph Algorithms
  - Reachability, connectivity, graph traversal
    - DFS and BFS
    - Typology of edges in a whatever-first-search tree
      - tree, forward, backward, cross
    - Post-ordering of nodes in a traversal
  - Topological orderings/Directed Acyclic Graphs (DAGs)
    - Reverse post-ordering is a topological ordering iff the graph is a DAG!
  - Shortest paths
    - Using BFS/DFS or Dijkstra (best-first-search)
    - Single-source vs. all-pairs
    - Betweenness centrality
  - Minimum Spanning Trees
    - Cut property and Cycle property
    - Boruvka: Add all safe edges across each cut, then recurse
    - Prim: Best first search: Repeatedly add T's safe edge to itself
- Network Flow
  - Max flow/min cut duality
  - Augmenting Paths and the residual graph
  - Ford-Fulkerson algorithm
  - Reduction to many other problems

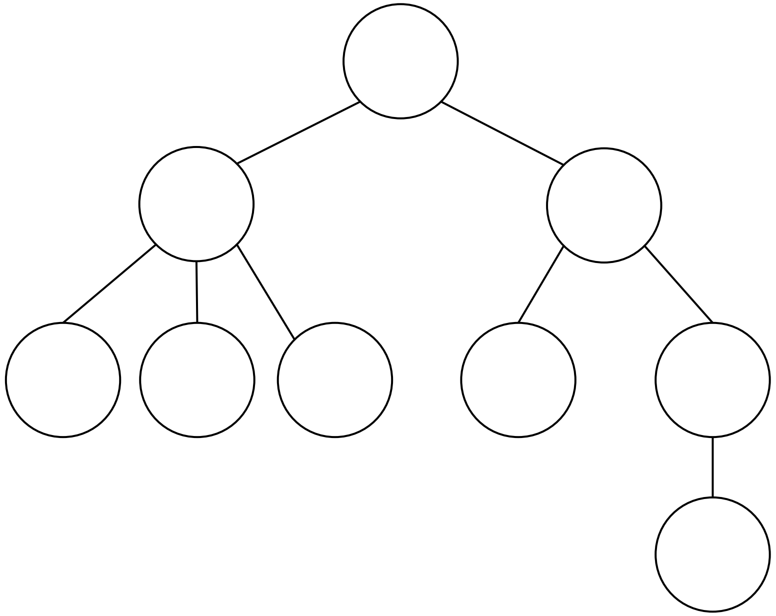
Graph Traversal

Breadth vs. Depth vs. Best (first search)

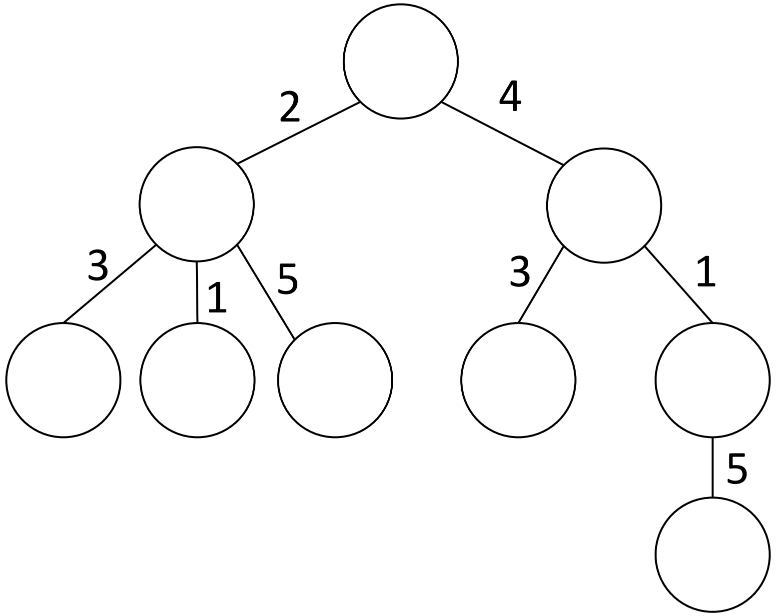
BFS



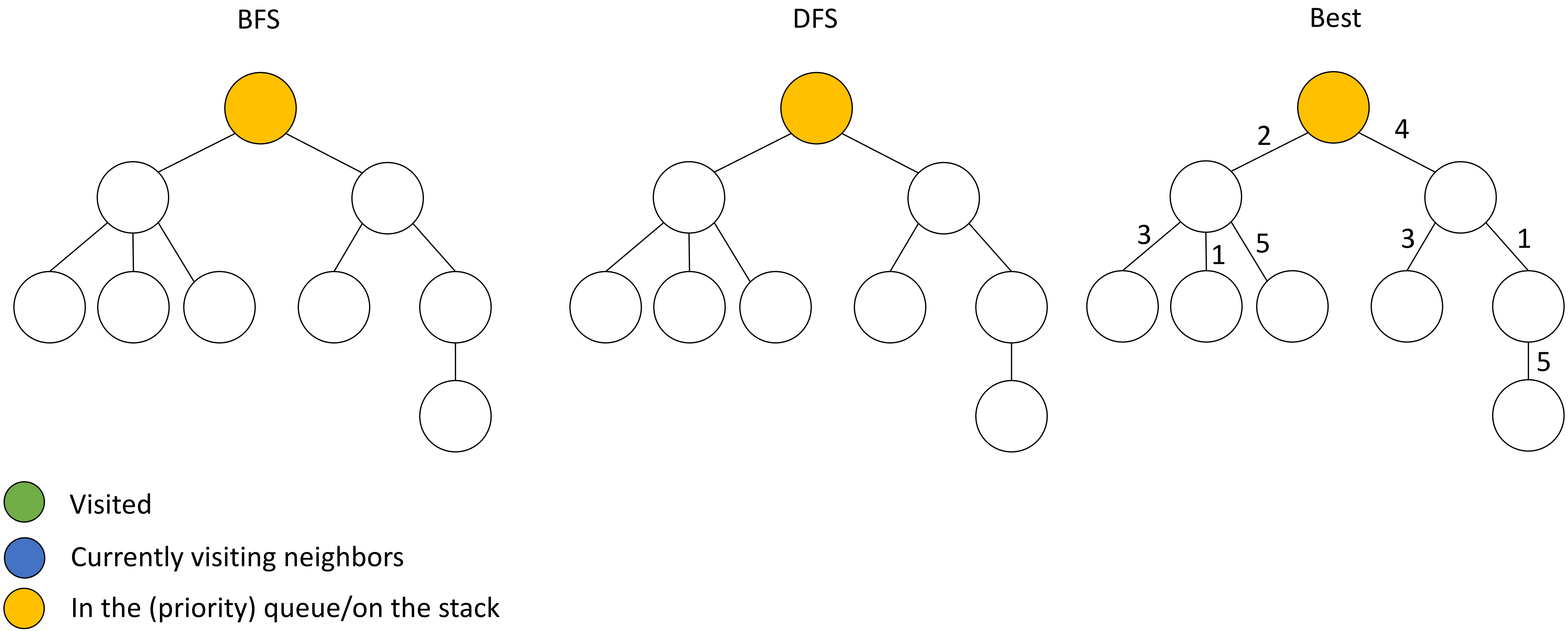
DFS



Best



Breadth vs. Depth vs. Best (first search)



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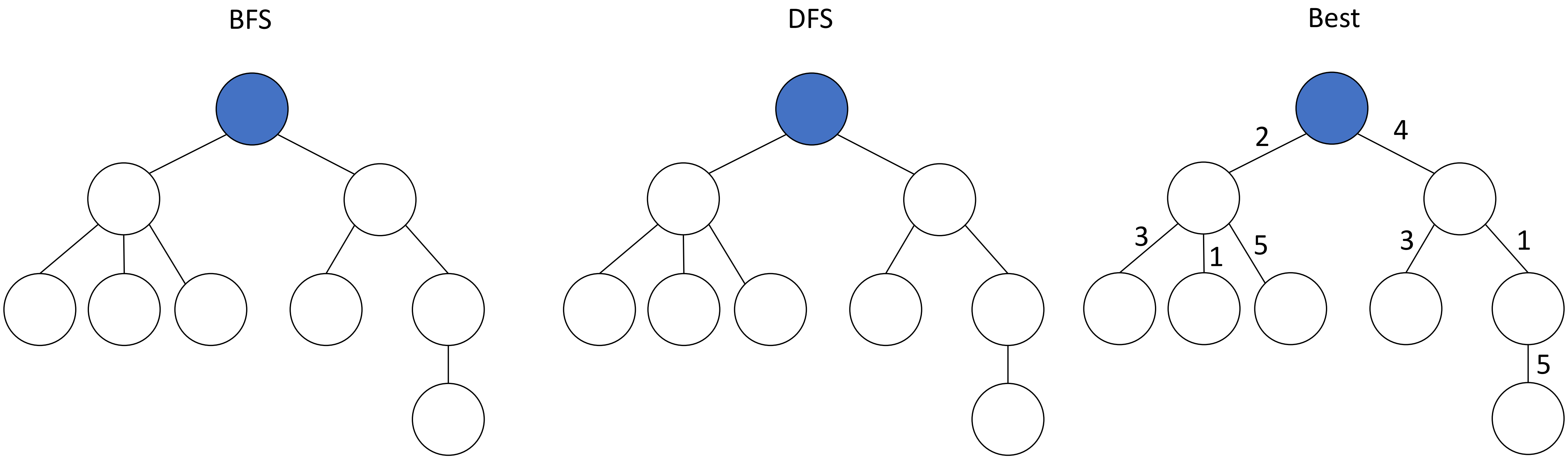
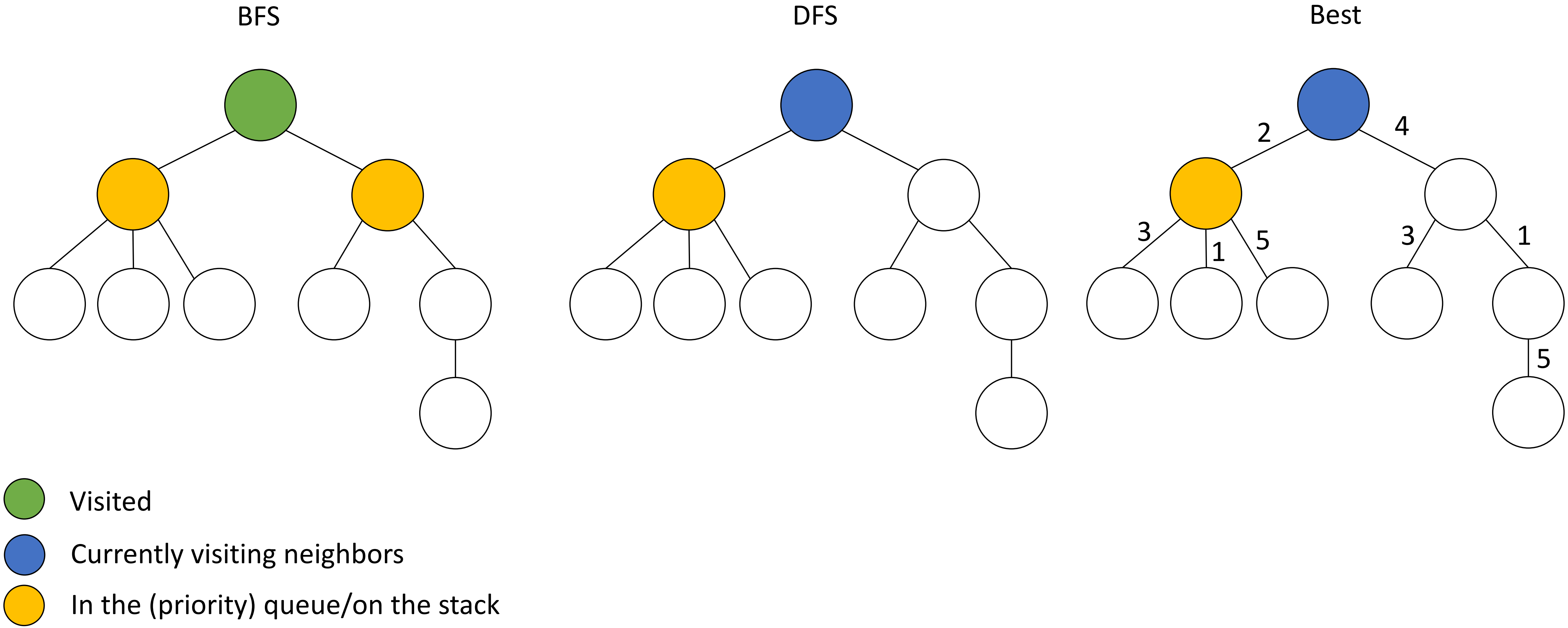


Visited

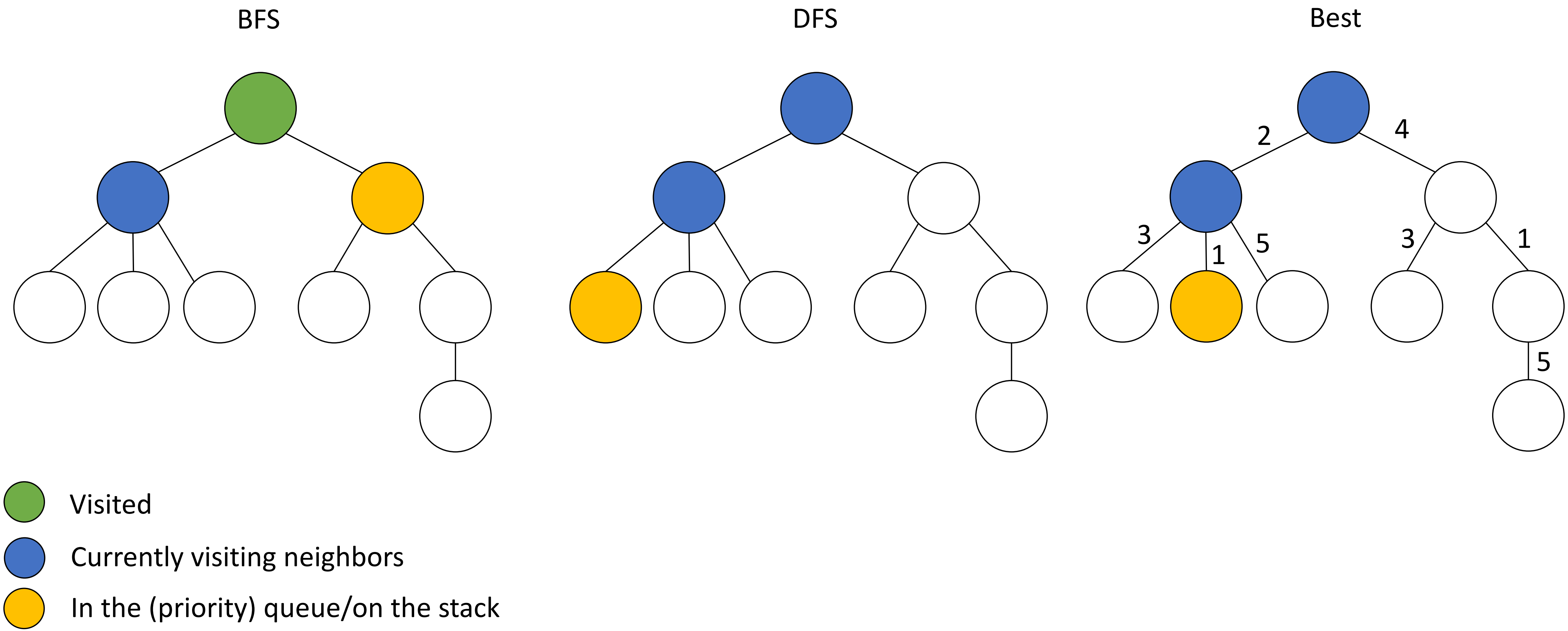
Currently visiting neighbors

In the (priority) queue/on the stack

Breadth vs. Depth vs. Best (first search)



Breadth vs. Depth vs. Best (first search)



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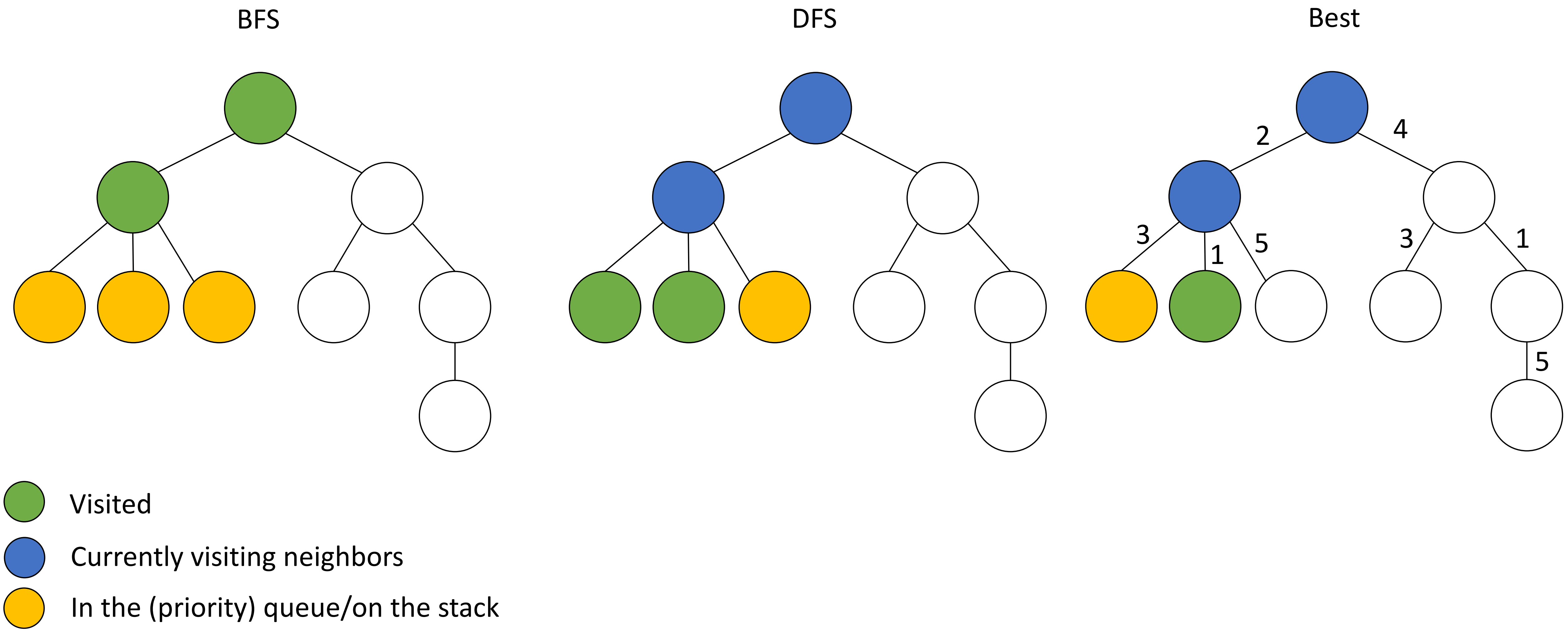
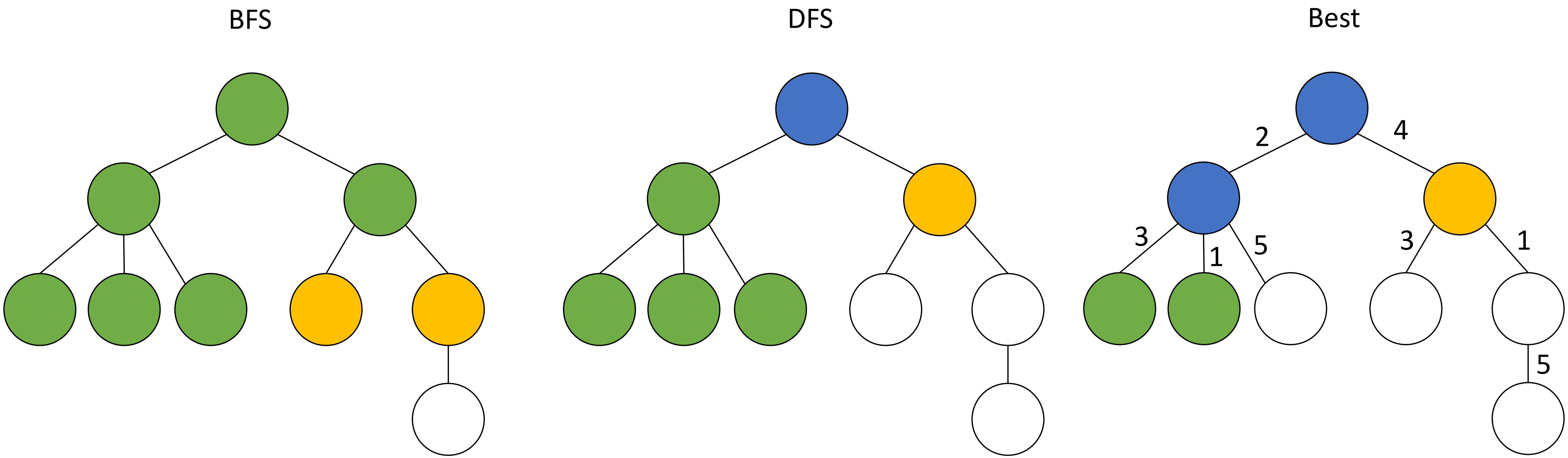


Breadth vs. Depth vs. Best (first search)



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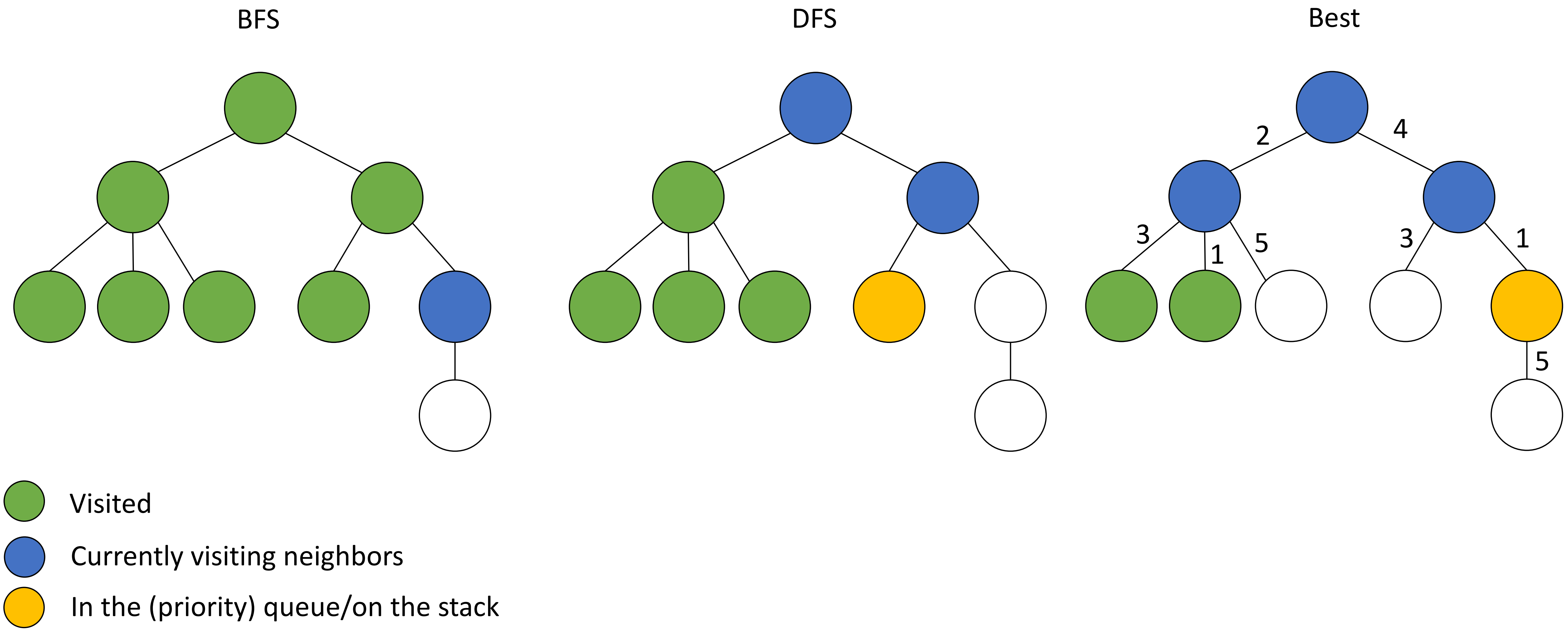

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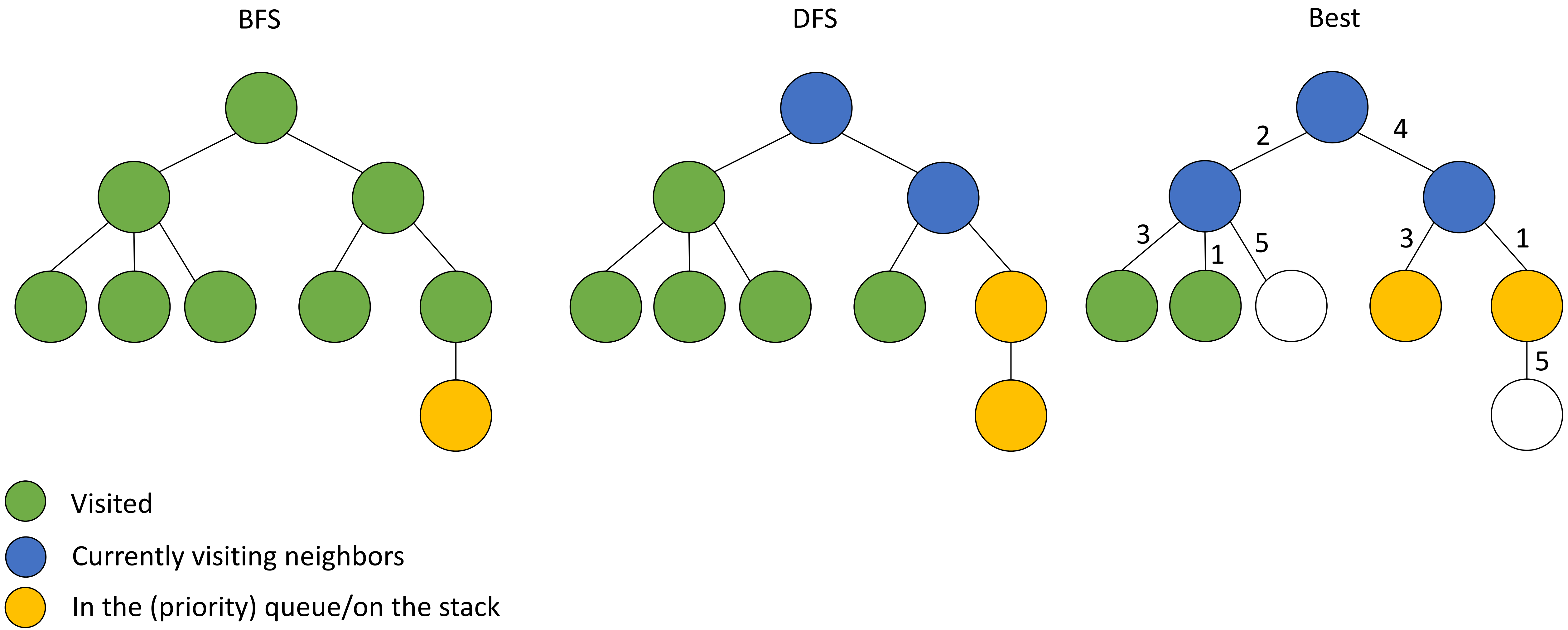
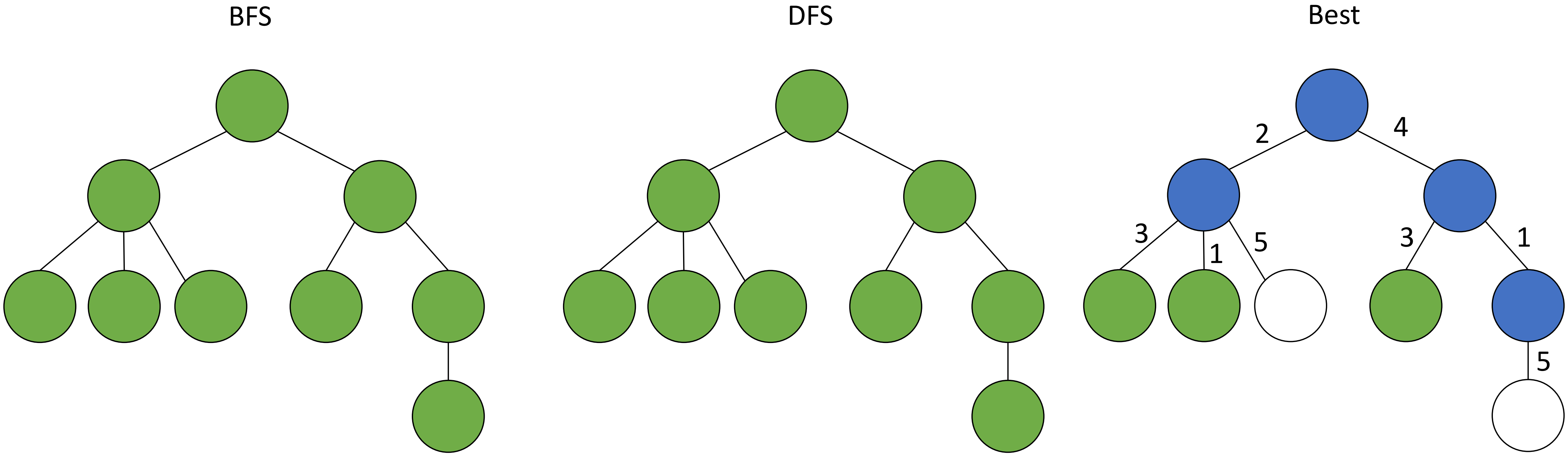


Breadth vs. Depth vs. Best (first search)



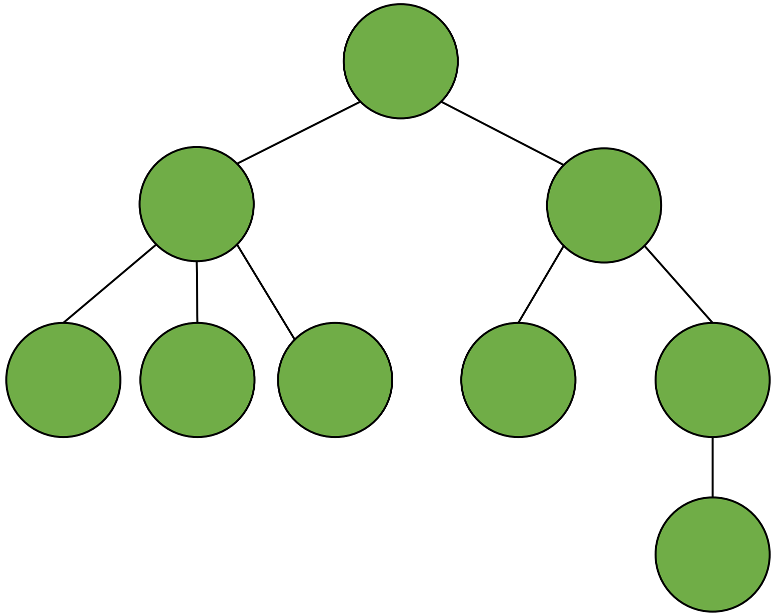
Visited

Currently visiting neighbors

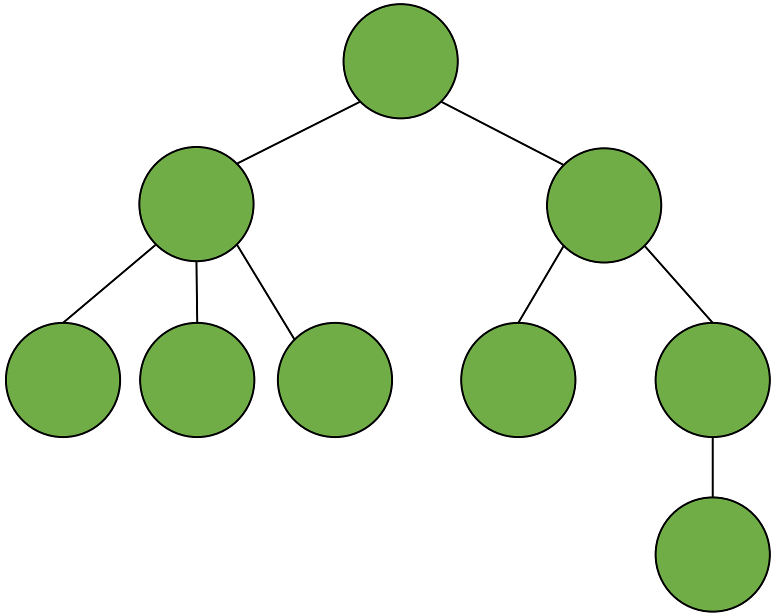
In the (priority) queue/on the stack

Breadth vs. Depth vs. Best (first search)

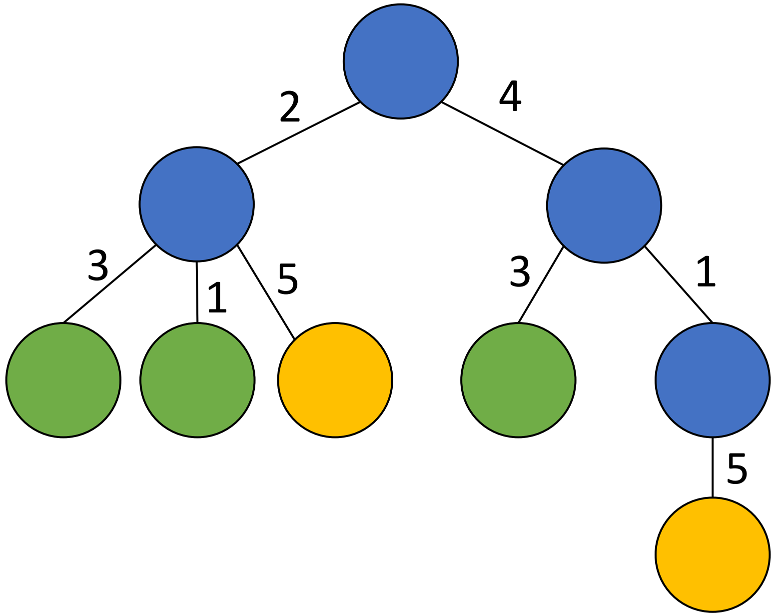
BFS

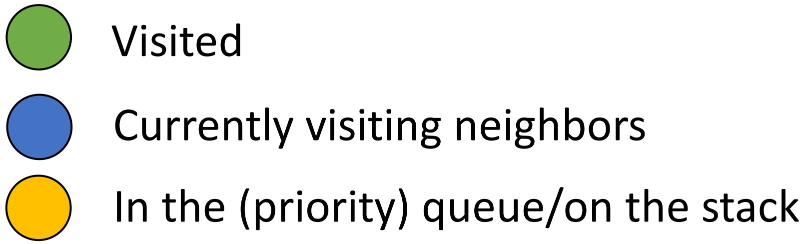


DFS

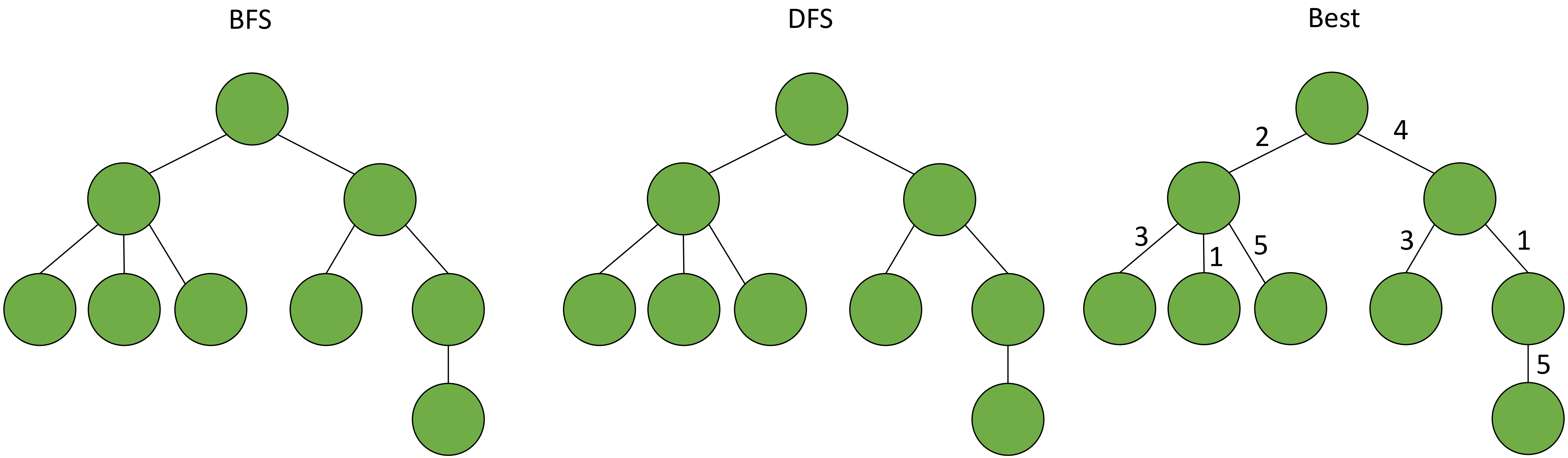


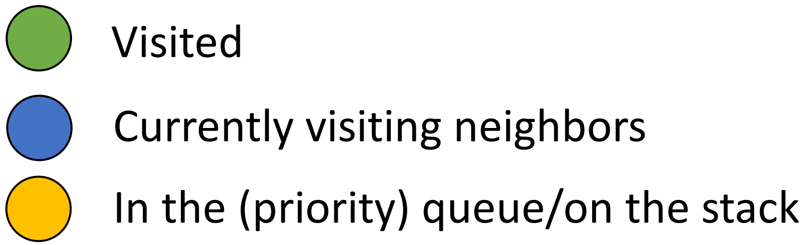
Best





Breadth vs. Depth vs. Best (first search)





Post-Ordering, DAGs, and Topological Ordering

Post-Ordering

A *post-ordering* of a graph $G = (V, E)$ is an ordering of the nodes based on "when" DFS from each node finished.

To get a post-order, we maintain a global clock variable that is initialized to 1.

Every time we finish calling DFS on all of a node's neighbors, we set its post-order value to the current value of clock, then increment clock.

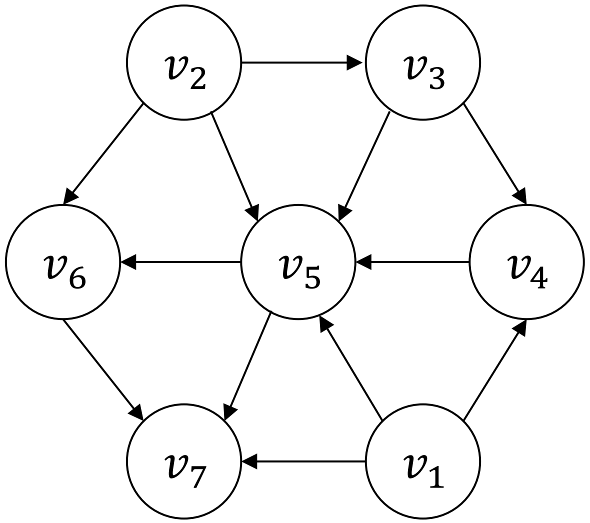
Recursive DFS with post-ordering

```
G = (V,E) is a graph
visited[u] = 0 for all u ∈ V
clock = 1

DFS(u):
  visited[u] = 1
  For v ∈ Neighbors(u):
    If visited[v] = 0:
      parent[v] = u
      DFS(v)

  post-visit(u)
```

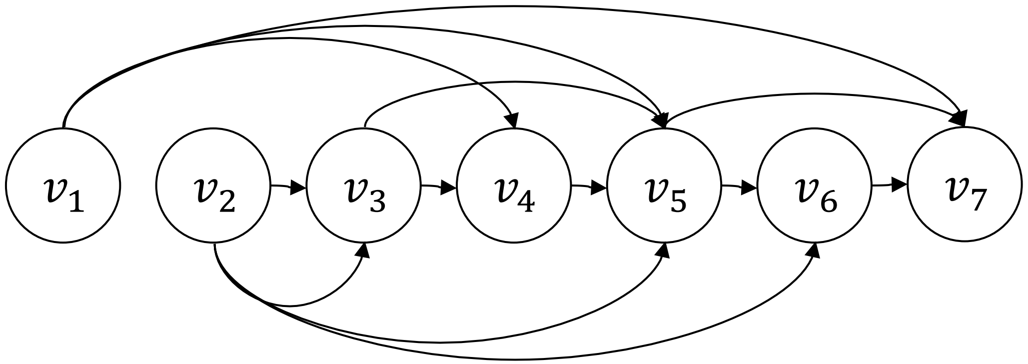
```
post-visit(u):
  set postorder[u] = clock
  clock ← clock + 1
```

Directed Acyclic Graph (DAG)

- A directed graph with no cycles
- Represent precedence relationships
  - “this” comes before “that”
  - “this” is prior to “that”



A *topological ordering* of a directed graph is a labeling of the nodes so that all edges point “forward”, meaning for all directed edges $(v_i, v_j), j > i$

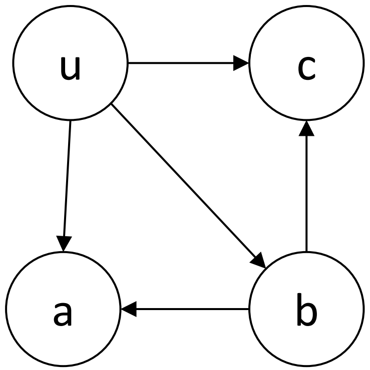


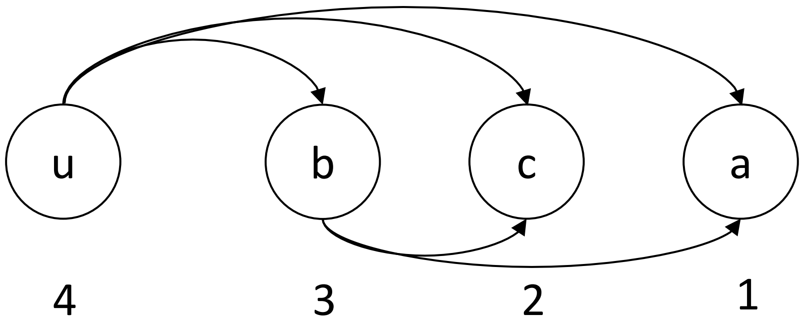
Key point: A reverse post-ordering of the nodes in a DAG is a topological ordering!

Topological Ordering

Ordering nodes by decreasing post-order gives a topological ordering.

Example:





Vertex	u	a	b	c
Postorder	4	1	3	2

Minimum Spanning Trees

Minimum Spanning Trees

A *spanning tree* is a set of edges $T \in E$ is a subgraph of a graph $G = (V, E)$ that (i) is a tree and (ii) contains all of the nodes $v \in V$.

A *minimum spanning tree* for a connected, weighted, undirected graph $G = (V, E, \{w_e\})$, where $w_e \in \mathbb{R}$ is a weight associated with each edge $e \in E$, is a spanning tree T with minimum weight $w(T)$:

$$w(T) = \sum_{e \in T} w_e$$

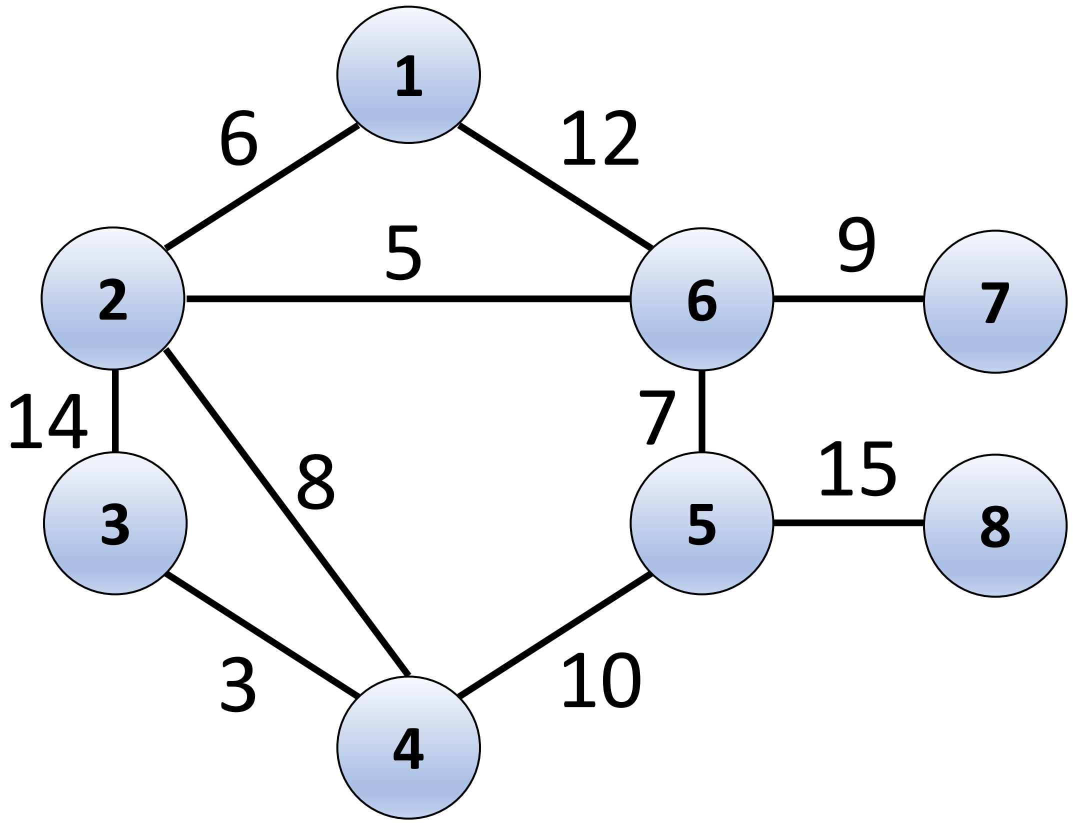
Borůvka's Algorithm

- **Borůvka:**
  - Let $T = \emptyset$
  - Repeat until T is connected:
    - Let $C_1, \dots, C_k$ be the connected components of (V, T)
    - Let $e_1, \dots, e_k$ be the safe edge for the cuts $C_1, \dots, C_m$
    - Add $e_1, \dots, e_k$ to T

- **Correctness:** every edge we add is safe

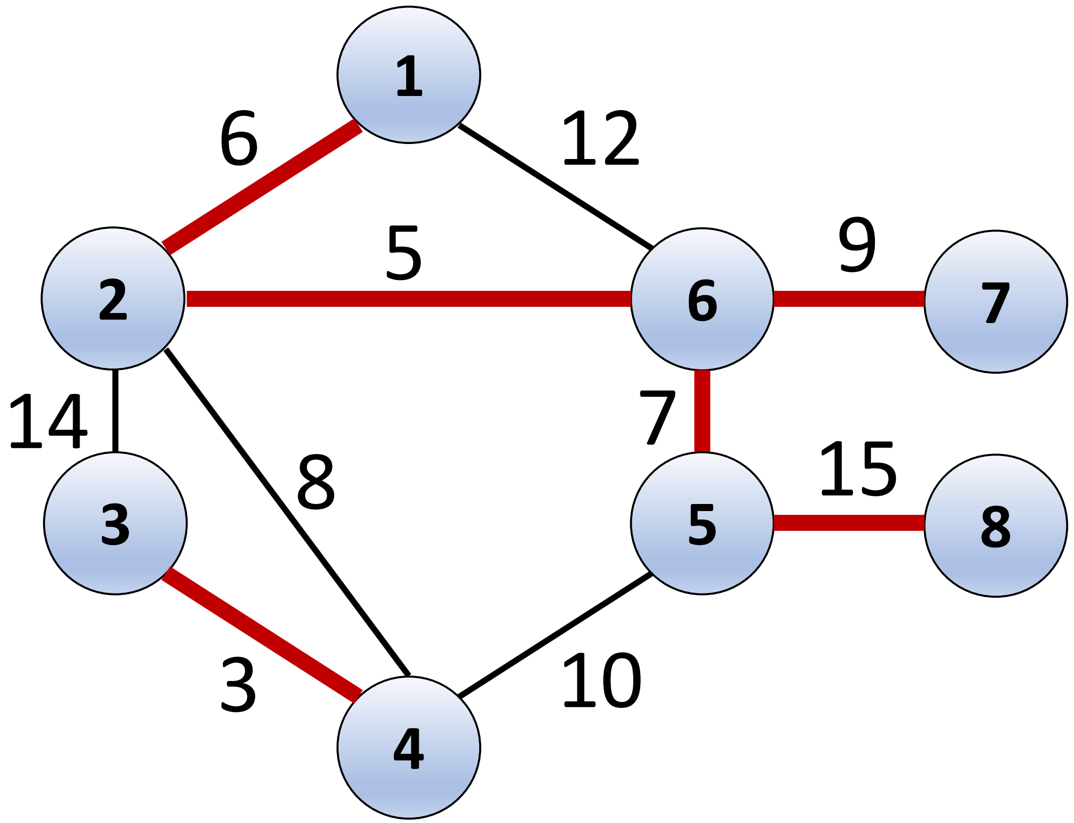
Borůvka's Algorithm

Label Connected Components



Borůvka's Algorithm

Add Safe Edges



Borůvka's Algorithm

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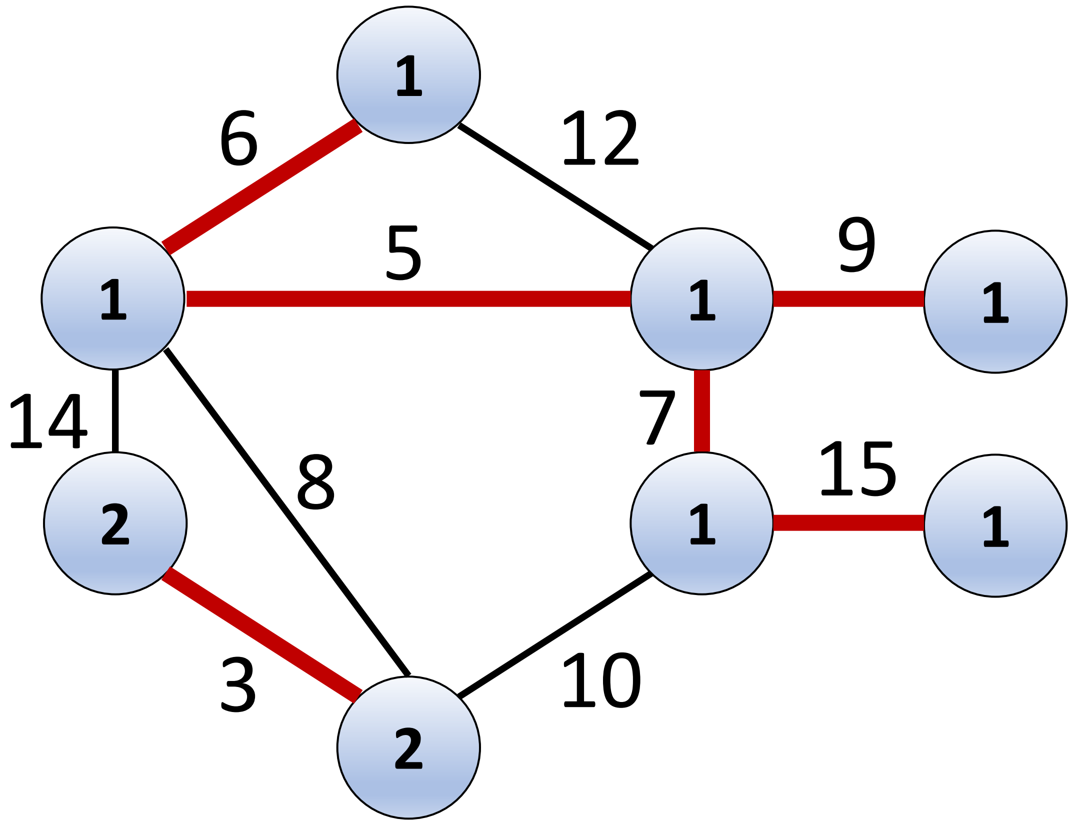
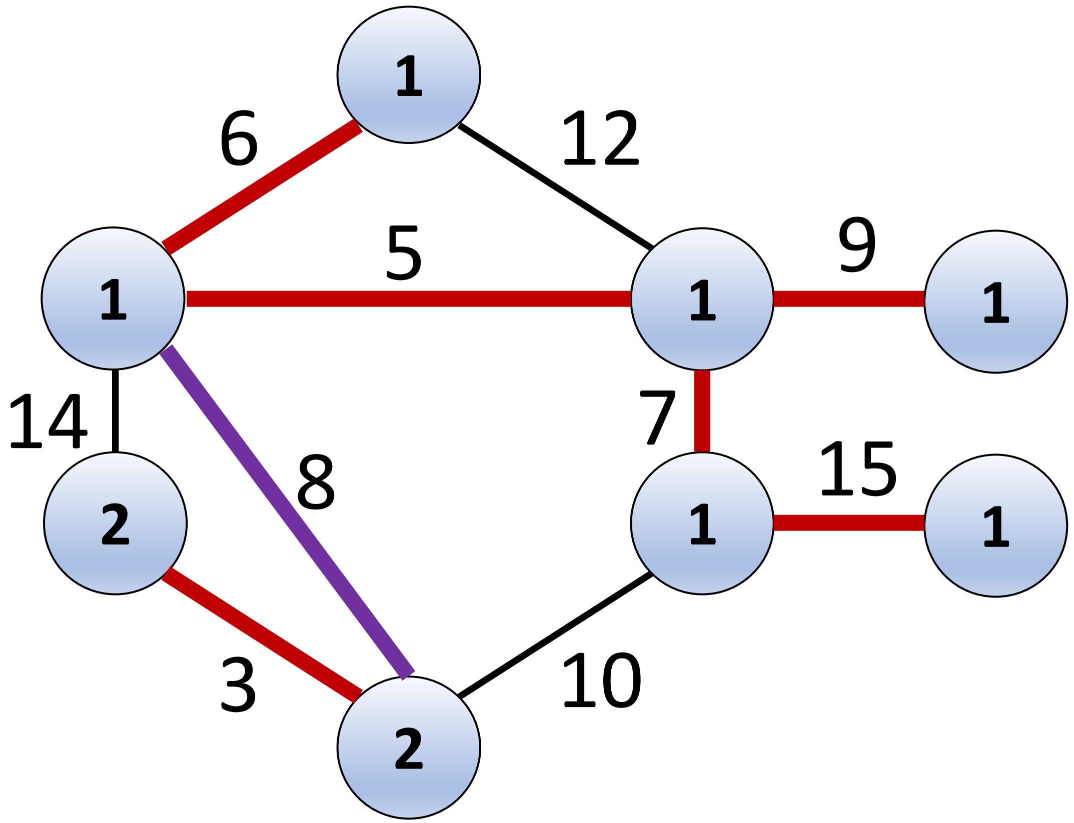


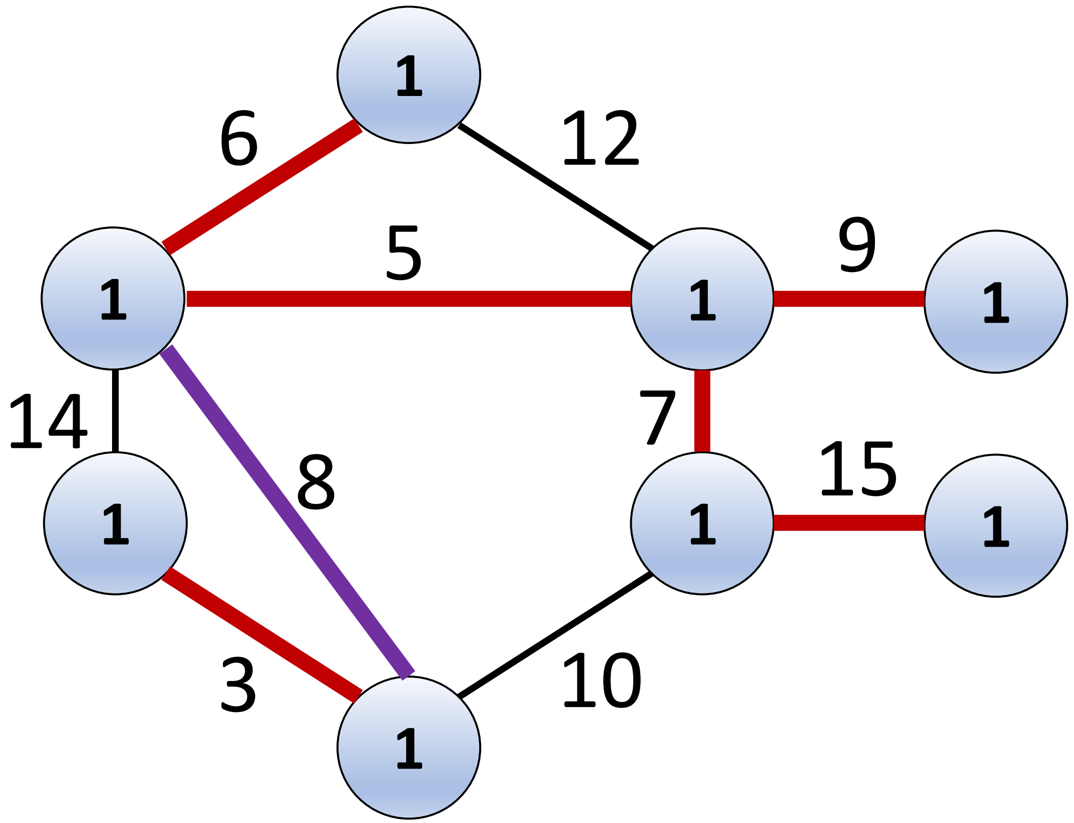
Borůvka's Algorithm

Add Safe Edges



Borůvka's Algorithm

Done!

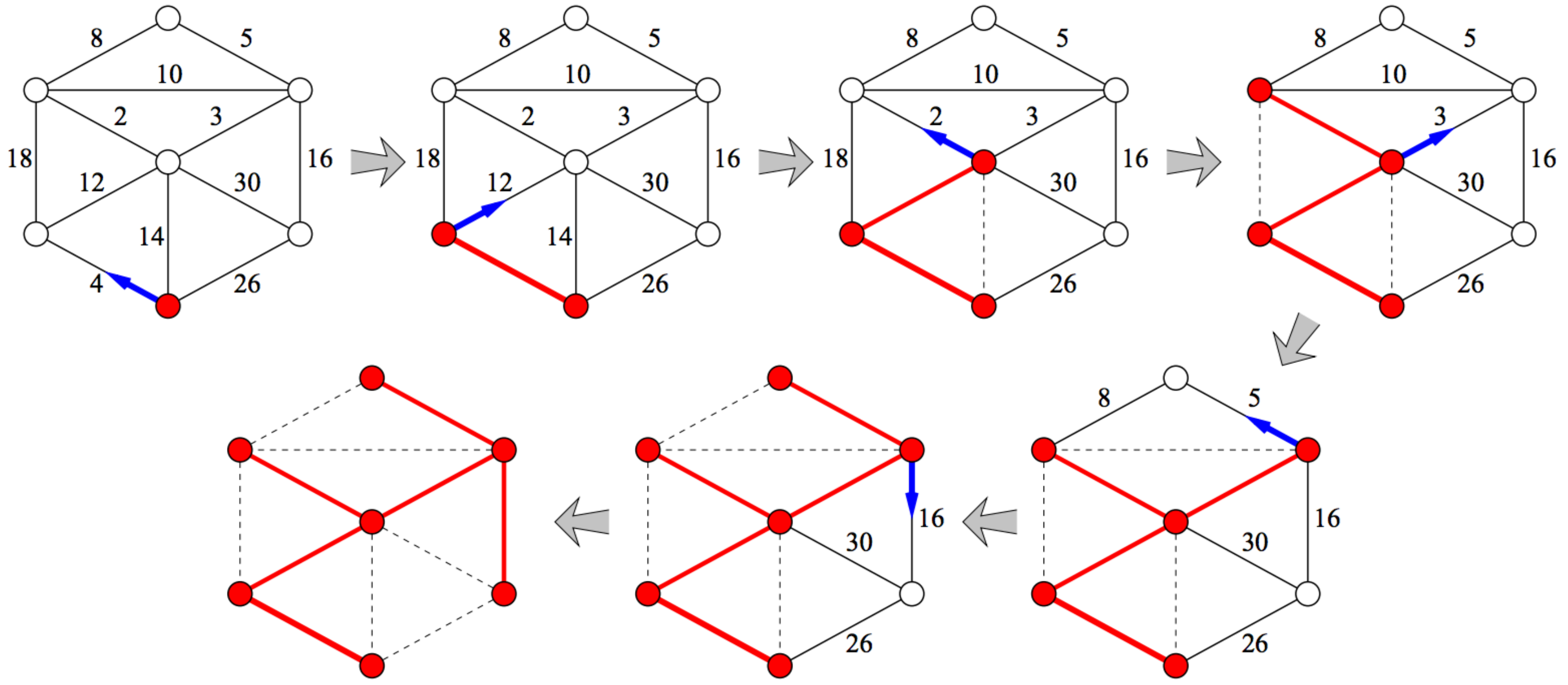


Prim's Algorithm

- **Prim Informal**
  - Let $T = \emptyset$
  - Let s be some arbitrary node and $S = \{s\}$
  - Repeat until $S = V$
    - Find the cheapest edge $e = (u, v)$ cut by S. Add e to T and add v to S

- **Correctness:** every edge we add is safe

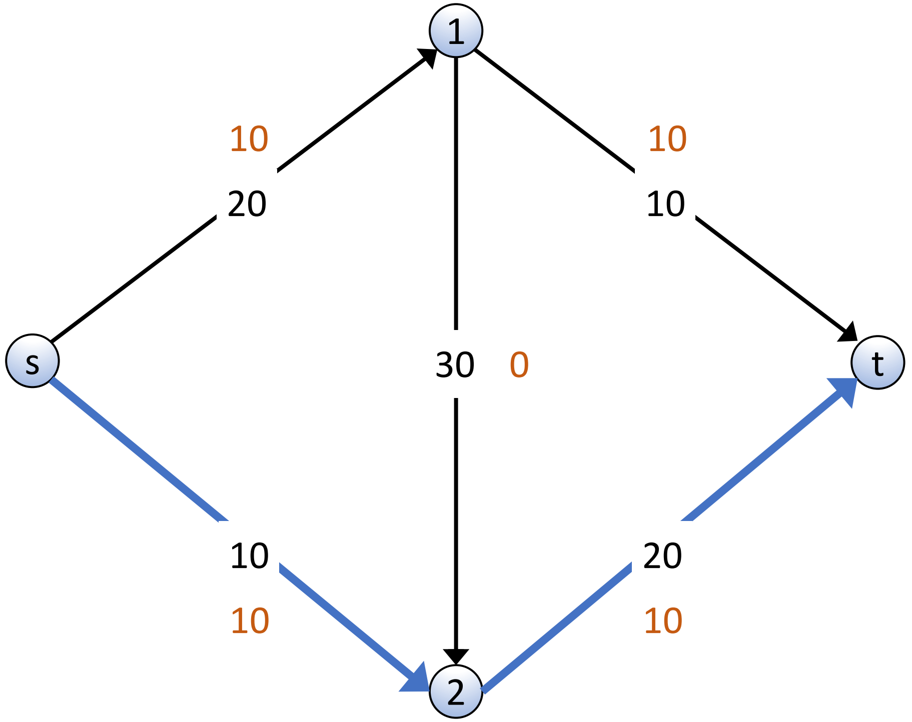
Prim’s Algorithm



Network Flow

Augmenting Paths

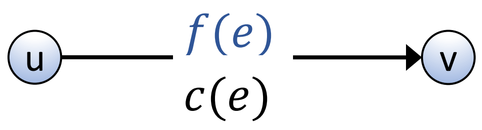
- Given a network $G = (V, E, s, t, \{c(e)\})$ and a flow f, an **augmenting path** P is an $s \to t$ path such that $f(e) < c(e)$ for every edge $e \in P$



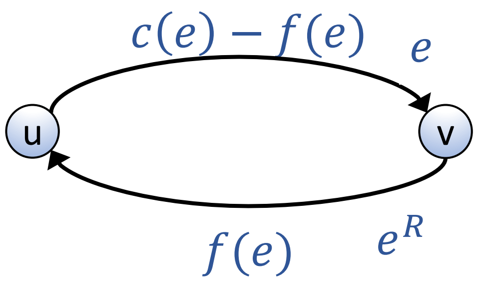
Adding uniform flow on an augmenting path results in a new valid s-t flow!

Residual Graphs

- Original edge: $e = (u, v) \in E$.
  - Flow $f(e)$, capacity $c(e)$



- Residual edge
  - Allows “undoing” flow
  - $e = (u, v)$ and $e^R = (v, u)$.
  - Residual capacity



- Residual graph $G_f = (V, E_f)$
  - Edges with positive residual capacity.
  - $E_f = \{e : f(e) < c(e)\} \cup \{e^R : c(e) > 0\}$.

Augmenting Paths in Residual Graphs

- Let G_f be a **residual graph**
- Let P be an augmenting path in the **residual graph**
- **Fact:** $f' = \text{Augment}(G_f, P)$ is a valid flow

```
Augment(G_f, P)
    b ← the minimum capacity of an edge in P
    for e ∈ P
        if e ∈ E:    f(e) ← f(e) + b
        else:        f(e) ← f(e) - b
    return f
```

Note: This is the same process as the recurrence in Erickson 10.3!

$$f'(u \to v) = \begin{cases} f(u \to v) + F & \text{if } u \to v \in P \\ f(u \to v) - F & \text{if } v \to u \in P \\ f(u \to v) & \text{otherwise} \end{cases}$$

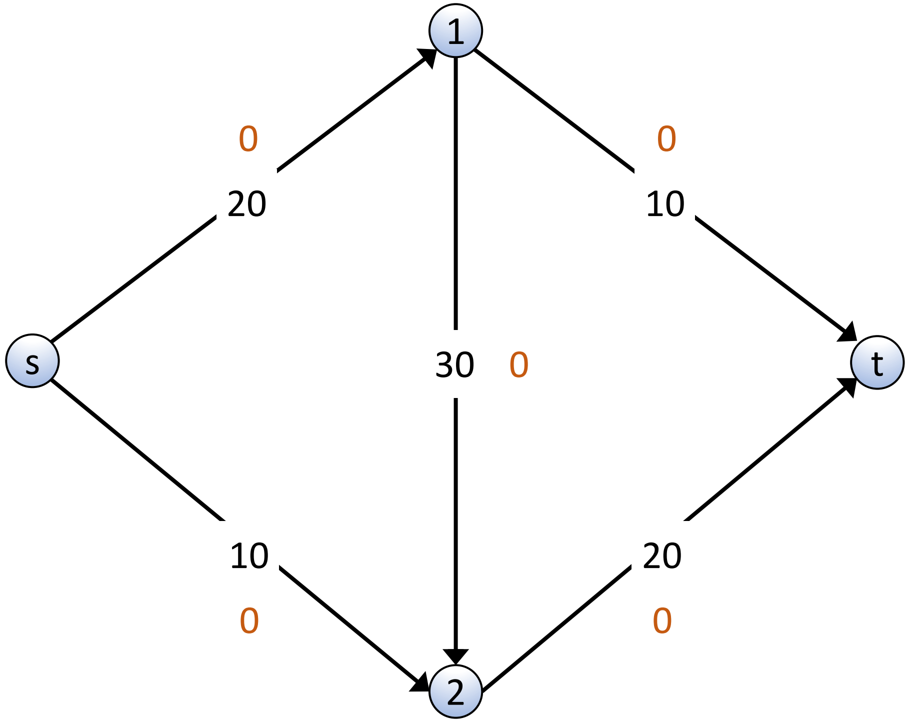
Ford-Fulkerson Algorithm

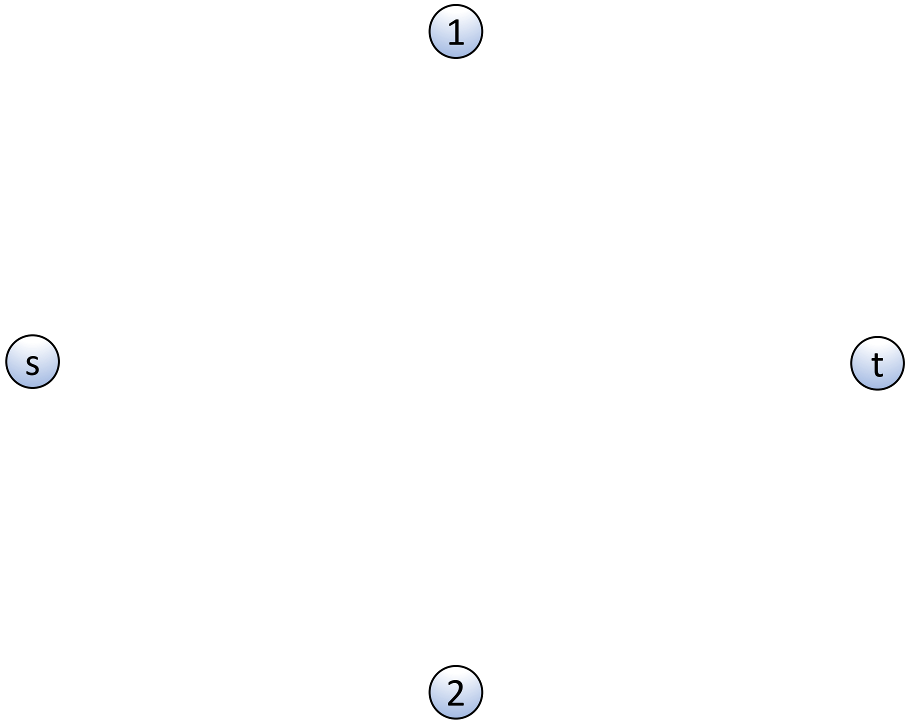
```
FordFulkerson(G,s,t,{c(e)})
    for e ∈ E: f(e) ← 0
    G_f is the residual graph

    while (there is an s-t path P in G_f)
        f ← Augment(G_f,P)
        update G_f

    return f
```

```
Augment(G_f, P)
    b ← the minimum capacity of an edge in P
    for e ∈ P
        if e ∈ E:    f(e) ← f(e) + b
        else:        f(e) ← f(e) - b
    return f
```

Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
- Repeat until you get stuck





Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
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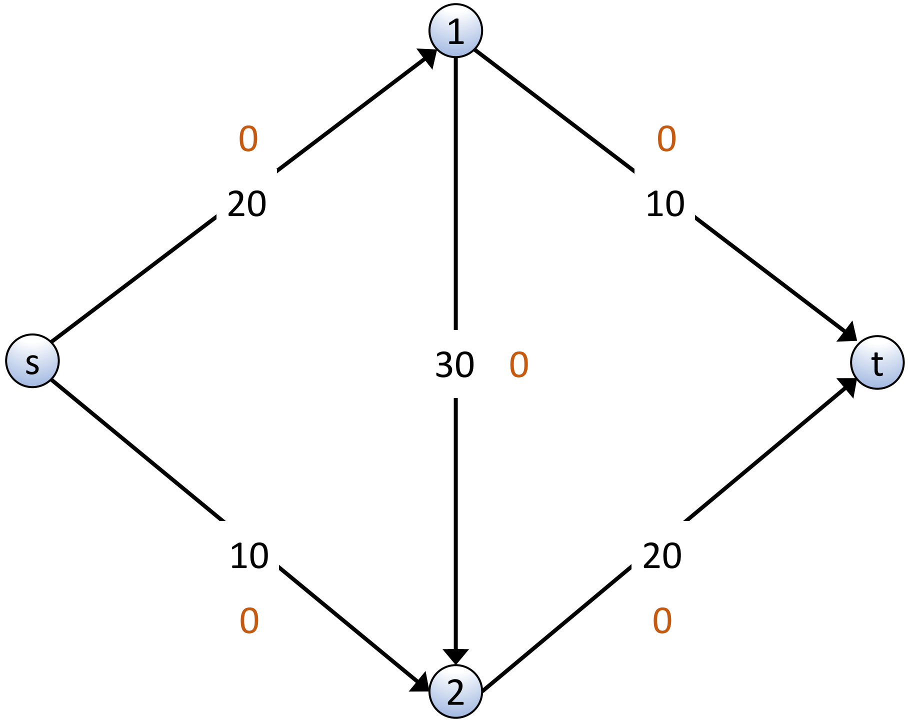

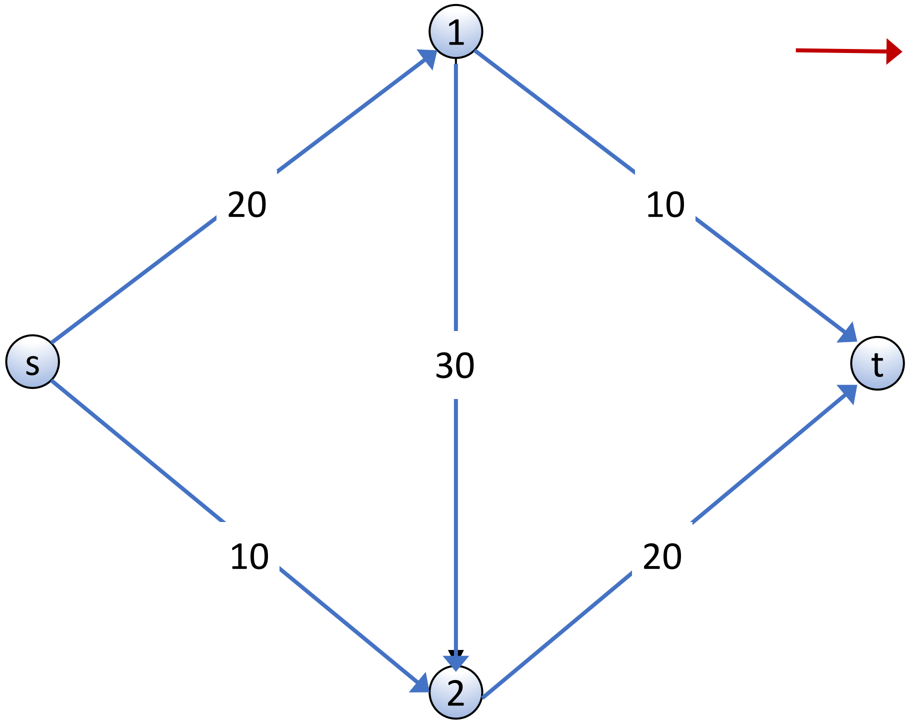

$e \in E$

$e^R \notin E$

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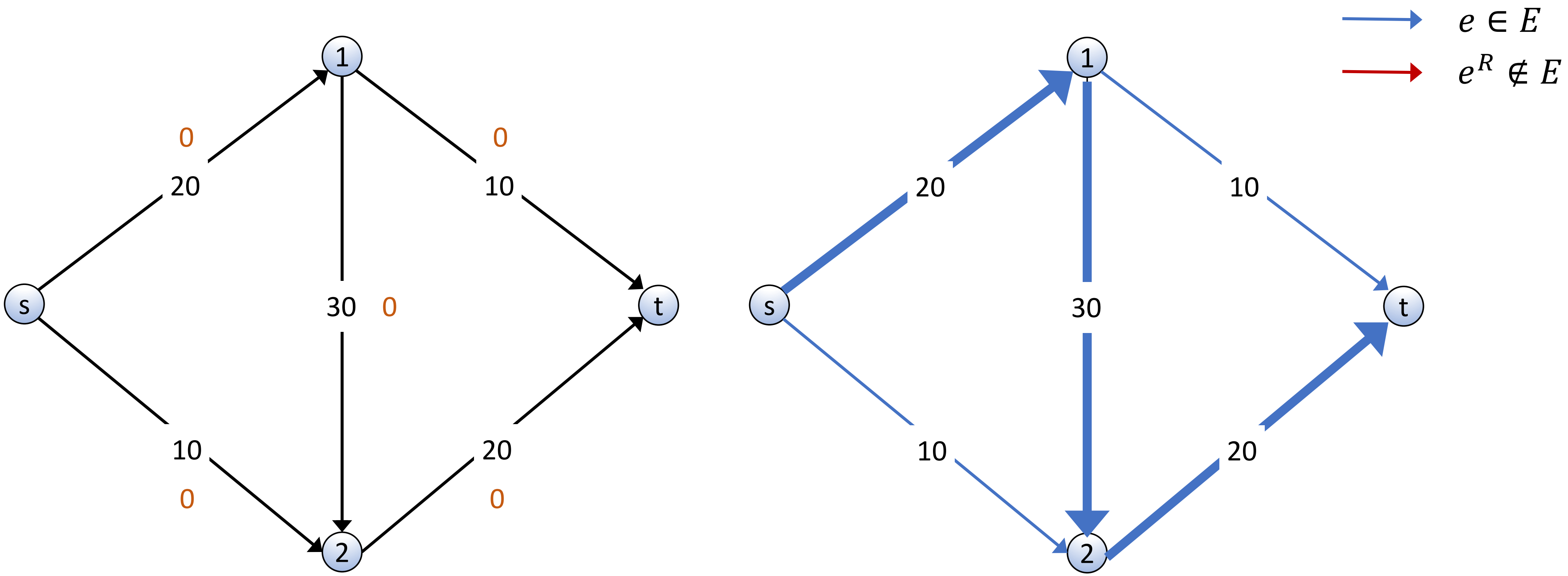

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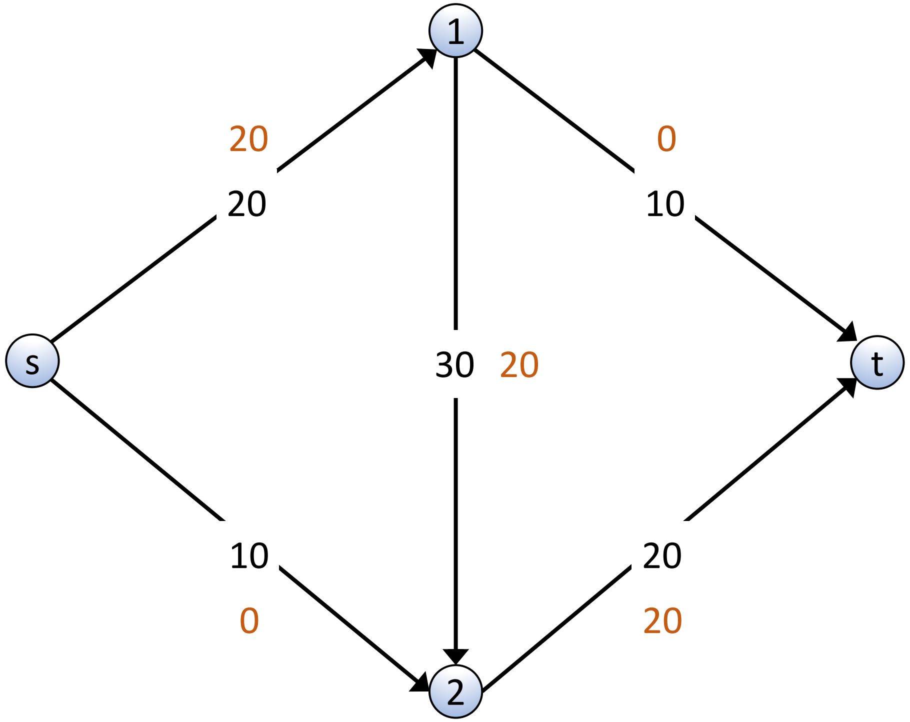

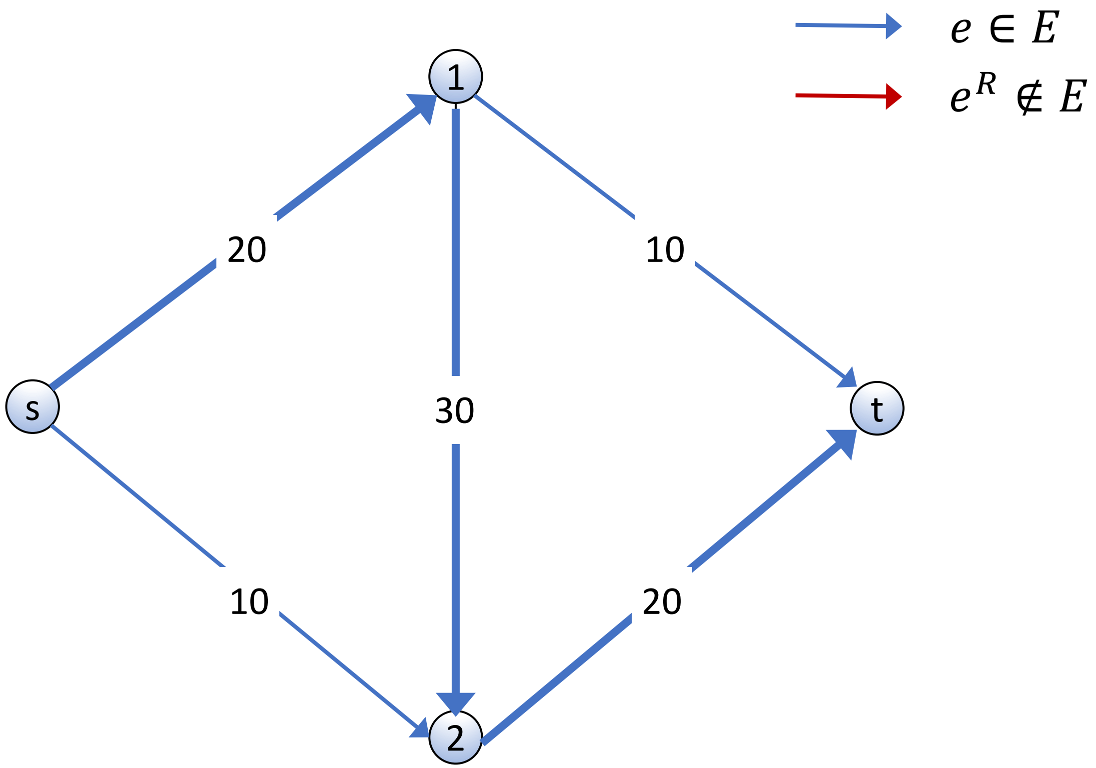

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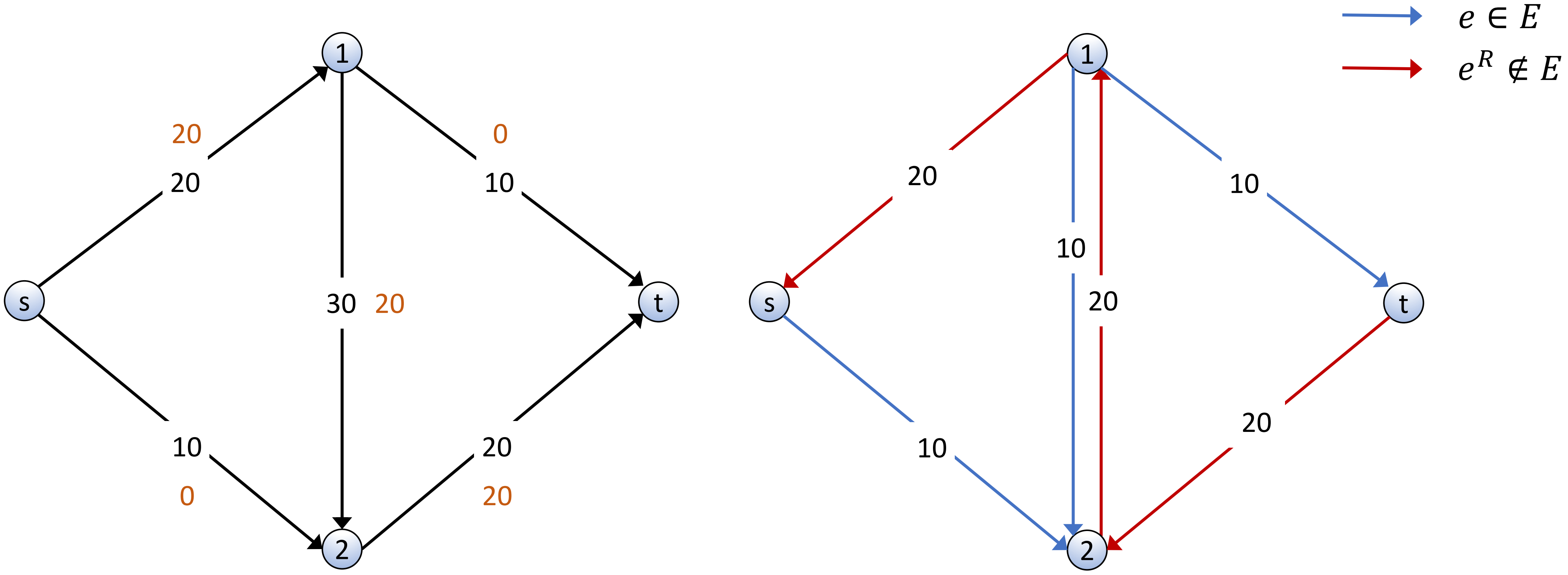

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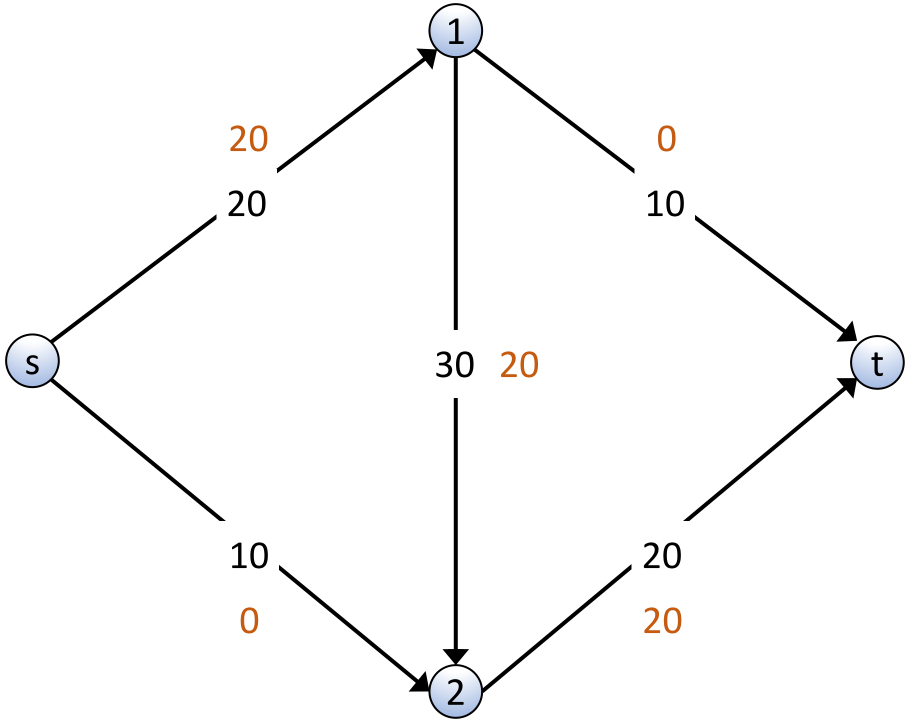

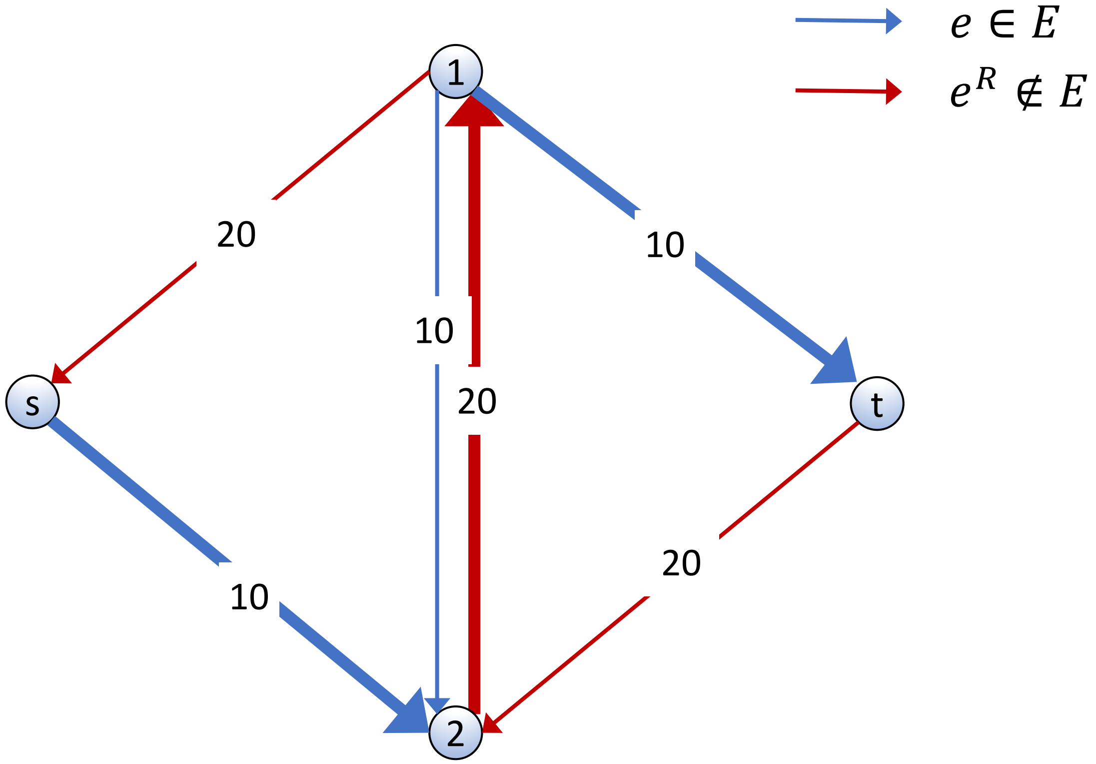

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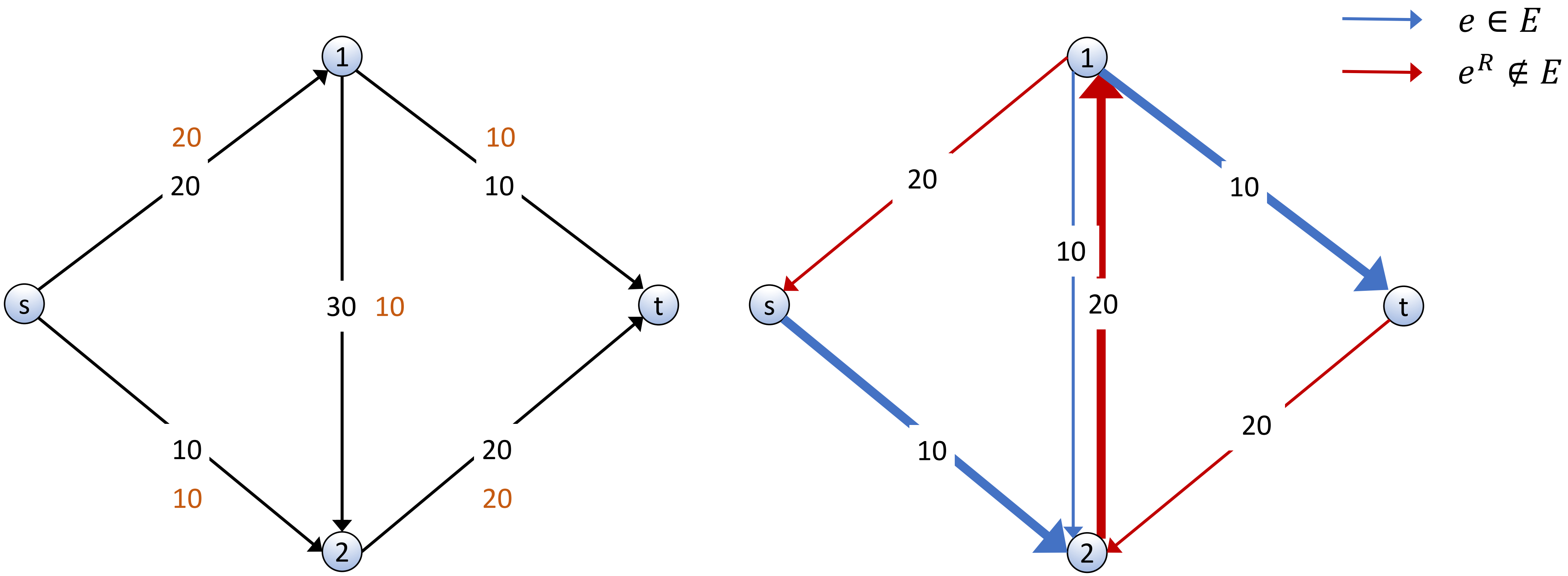

Ford-Fulkerson Algorithm

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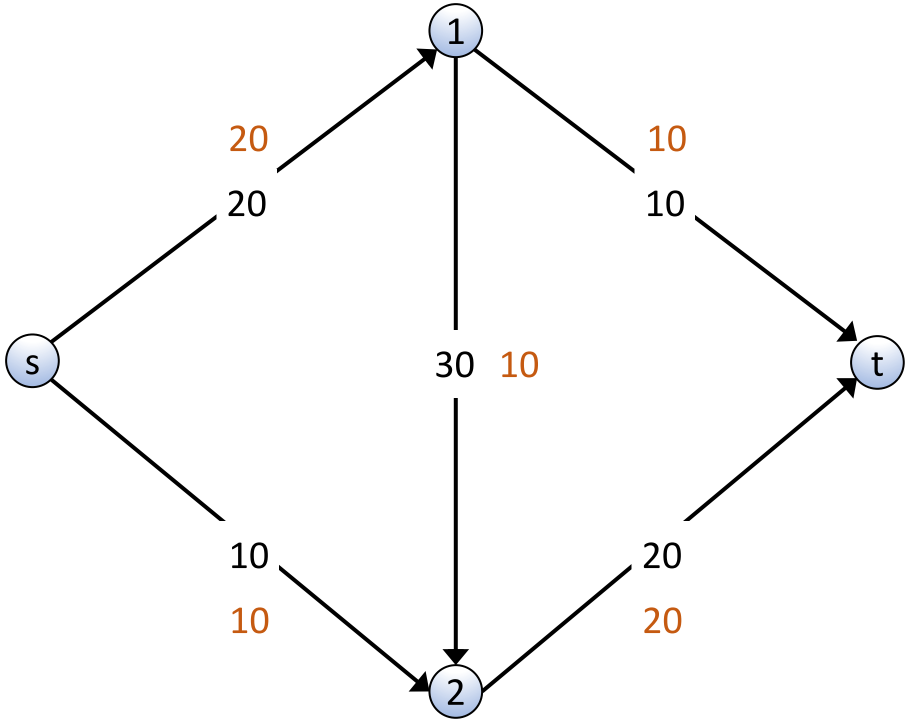

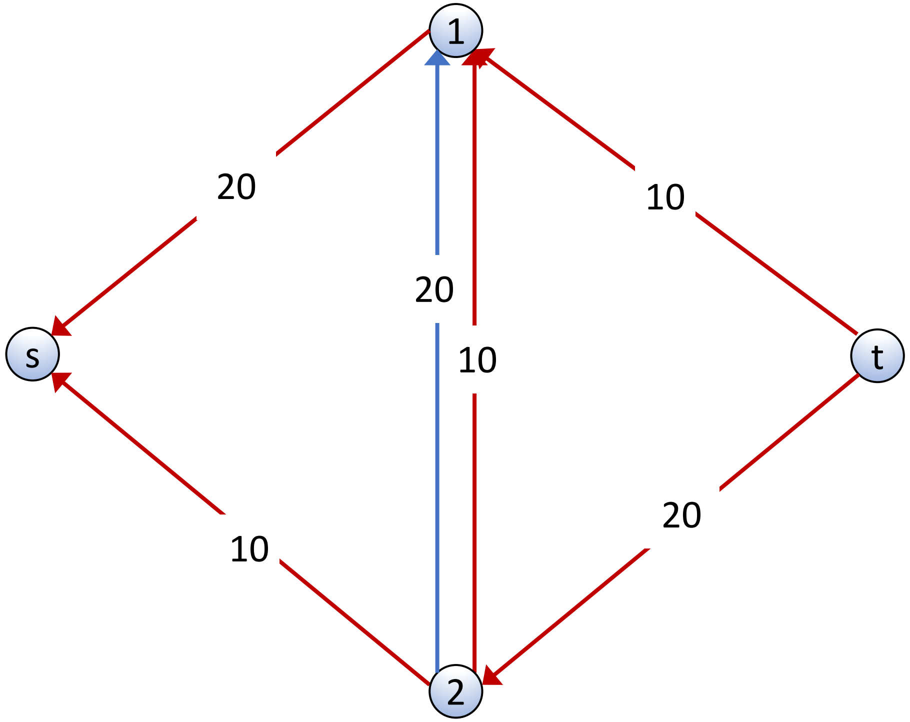

Network Flow Summary

- **The Ford-Fulkerson Algorithm solves maximum s-t flow**
  - Running time $O\left(m \cdot val(f^*)\right)$ in networks with integer capacities

- **Strong MaxFlow-MinCut Duality: max flow = min cut**
  - The value of the maximum s-t flow equals the capacity of the minimum s-t cut
  - If f^* is a maximum s-t flow, then the set of nodes reachable from s in G_{f^*} gives a minimum cut
  - Given a max-flow, can find a min-cut in time $O(n + m)$

- **Every graph with integer capacities has an integer maximum flow**
  - Ford-Fulkerson will return an integer maximum flow

More questions?

Wrap-up

No class tomorrow!

Homework 5 due tonight, solutions out tomorrow morning

- Get in touch ASAP (not 10PM) if you need more time!

Midterm 2 released Wednesday 8PM and due Friday 8PM Boston time!