

Lecture 18: Greedy Algorithms + Midterm Review

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bit.ly/cs3000syllabus

Business

Homework 5 due tonight at midnight Boston time, solutions will be released tomorrow morning

No class tomorrow, midterm review moved to today

Extra credit assignment available as of yesterday

- Optional
- 6 points on the final exam
- Available until Sunday June 21st

Midterm 2 to be released tomorrow night, due Friday night

- Topics: Graph algorithms and network flow

Greedy Algorithms

- For some problems, we can think of simple decision making rules that intuitively guide us towards a solution
 - Best-first search: We want to find shortest paths/minimum trees, so only choose edges that can be included in these solutions!
- Applying this idea does not always work as intended!
 - Maximum flow: We tried assigning flow based on best-first search, but we showed that the algorithm will get stuck if it is not able to modify the flow!
- Algorithms that rely on repeatedly making optimal *local decisions* to eventually reach an optimal global solution are called *greedy algorithms*

Example: Files on Tape

Before any of us were born, computers used to exist on magnetic tape.

Imagine we have such a tape, split in to segments we will call “blocks”, where each block contains data from a single file. Each file is referred to by an integer index i , and has length in blocks $L[i]$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 |
|---|---|---|---|---|---|---|---|---|---|

To read file k , the tape head needs to first skip all of the files before k . Therefore, the *cost* of accessing file k can be written as

$$\text{cost}(k) = \sum_{i=1}^k L[i]$$

$$\begin{aligned} k &= 3 \\ L[1] + L[2] + L[3] \\ &= 3 + 2 + 3 = 8 \end{aligned}$$

Example: Files on Tape

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 |
|---|---|---|---|---|---|---|---|---|---|

Assuming all files are equally likely to be accessed, we can write the *expected* (equivalently, average) cost of accessing file k as

$$\mathbb{E}[\text{cost}] = \frac{1}{n} \sum_{i=1}^n \text{cost}(i) = \frac{1}{n} \sum_{k=1}^n \sum_{i=1}^k L[i]$$

n is # of files

Example: Files on Tape

$$\begin{aligned}\mathbb{E}[\text{cost}] &= \frac{1}{4} \cdot (\text{cost}(1) + \text{cost}(2) + \text{cost}(3) + \text{cost}(4)) \\ &= \frac{1}{4} \cdot (3 + 5 + 8 + 10) = \frac{26}{4}\end{aligned}$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 |
|---|---|---|---|---|---|---|---|---|---|

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What order should we keep the files in?

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|----|-----------------------------------|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\mathbb{E}[cost] = \frac{26}{4}$ |
| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | |

We can modify the order of the files on the tape, resulting in a permutation π where $\pi(i)$ returns the index of the file in the i th block. We can then rewrite the *expected* (average) cost of accessing file k as

$$\mathbb{E}[cost(\pi)] = \frac{1}{n} \sum_{k=1}^n \sum_{i=1}^k L[\pi(i)]$$

$$\begin{aligned}\pi(8) &= 3 \\ \pi(5) &= 2\end{aligned}$$

Intuitively: To minimize average cost, we should store the smallest files first, otherwise we will need to unnecessarily spend time skipping the large files to read smaller ones!

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 4 | 4 | 1 | 1 | 1 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|

What order should we keep the files in?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 |
|---|---|---|---|---|---|---|---|---|---|

 $\mathbb{E}[cost] = \frac{26}{4}$

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 4 | 4 | 1 | 1 | 1 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|---|---|

4 4 2 2

$$\mathbb{E}[cost(\pi)] = \frac{2 + 4 + 7 + 10}{4} = \frac{23}{4}$$

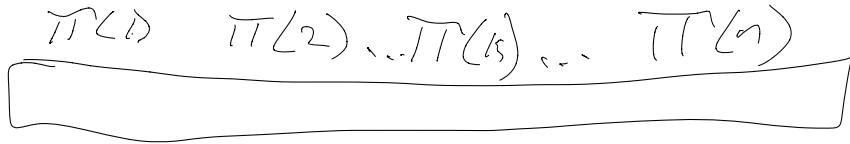
Greedy Algorithm for Storing Files

Input: A set of files labeled $1 \dots n$ with lengths $L[i]$

Output: An ordering of the files on the tape s.t. $E[\text{cost}]$ is minimized

Repeat until all files are on the tape:

1. Find the unwritten file with minimum length (break ties arbitrarily)
2. Write that file to the tape



$$L[1] \leq L[2] \dots \leq L[k] \dots \leq L[n]$$

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Repeat until all files are on the tape:

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How can we show this is optimal?

Proof of optimality

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 | 4 |
|---|---|---|---|---|---|---|---|---|---|

Claim: $\mathbb{E}[\text{cost}(\pi)]$ is minimized when $L[\pi(i)] \leq L[\pi(i + 1)]$ for all i .

Proof of optimality

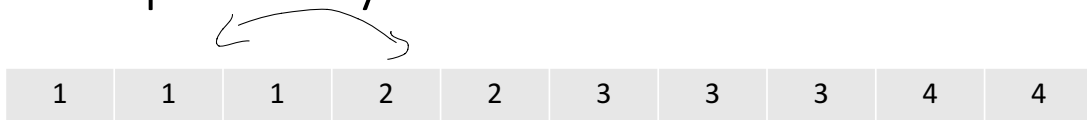
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Proof:

Let $a = \pi(i)$ and $b = \pi(i + 1)$ and suppose $L[a] > L[b]$ for some index i .

Proof of optimality



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Let $a = \pi(i)$ and $b = \pi(i + 1)$ and suppose $L[a] > L[b]$ for some index i .

If we swap the files a and b on the tape, then the cost of accessing a increases by $L[b]$ and the cost of accessing b decreases by $L[a]$.

(2 2 1 1 ...)

Proof of optimality

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Overall, the swap changes the expected cost by $\frac{L[b]-L[a]}{n}$.

Proof of optimality

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This change represents an improvement because $L[b] < L[a]$.

Proof of optimality

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This change represents an improvement because $L[b] < L[a]$.

Average cost for example above: $\frac{26}{4}$

Average cost after swapping files 1 and 2: $\frac{1}{4}(2 + 5 + 8 + 10) = \frac{25}{4}$

$$\frac{26}{4} + \frac{2-3}{4} = \frac{26-1}{4} = \frac{25}{4}$$

Proof of optimality

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If we swap the files a and b on the tape, then the cost of accessing a increases by $L[b]$ and the cost of accessing b decreases by $L[a]$.

Overall, the swap changes the expected cost by $\frac{L[b]-L[a]}{n}$.

This change represents an improvement because $L[b] < L[a]$.

Thus, if the files are out of length-order, we can decrease expected cost by swapping pairs to put them in order.

Wrap-up

Greedy algorithms repeatedly apply a simple rule to eventually find an optimal solution

Inductive Exchange Arguments are strategies for proving correctness of some greedy algorithms

Next Week:

- Data Compression with Huffman Codes

- Proof strategies for greedy algorithms

 - Inductive exchange

 - Greedy-stays-ahead

Midterm 2 Review/Q&A

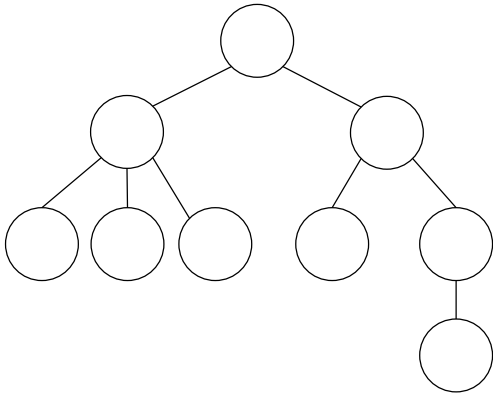
Topics

- Graph Algorithms
 - Reachability, connectivity, graph traversal
 - DFS and BFS
 - Typology of edges in a whatever-first-search tree
 - tree, forward, backward, cross
 - Post-ordering of nodes in a traversal
 - Topological orderings/Directed Acyclic Graphs (DAGs)
 - Reverse post-ordering is a topological ordering iff the graph is a DAG!
 - Shortest paths
 - Using BFS/DFS or Dijkstra (best-first-search)
 - Single-source vs. all-pairs
 - Betweenness centrality
 - Minimum Spanning Trees
 - Cut property and Cycle property
 - Boruvka: Add all safe edges across each cut, then recurse
 - Prim: Best first search: Repeatedly add T 's safe edge to itself
- Network Flow
 - Max flow/min cut duality
 - Augmenting Paths and the residual graph
 - Ford-Fulkerson algorithm
 - Reduction to many other problems

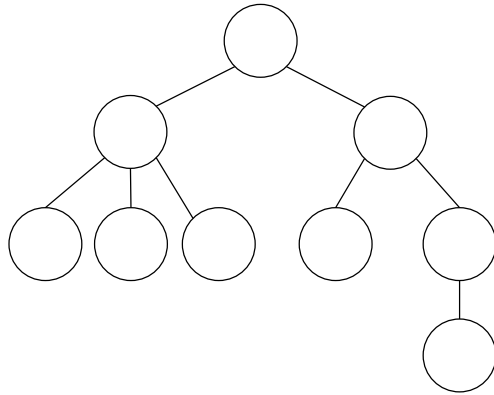
Graph Traversal

Breadth vs. Depth vs. Best (first search)

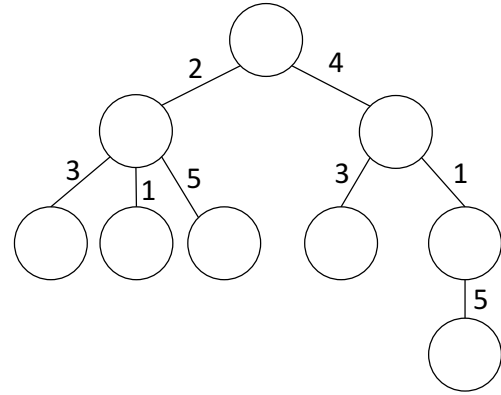
BFS



DFS

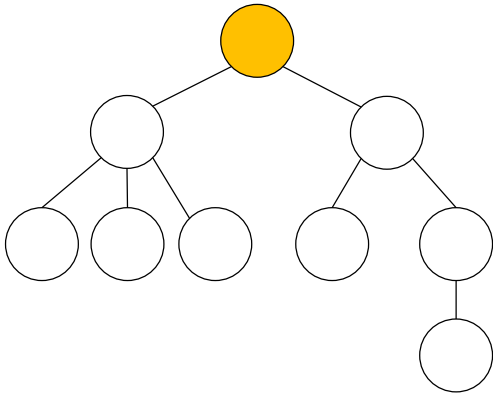


Best

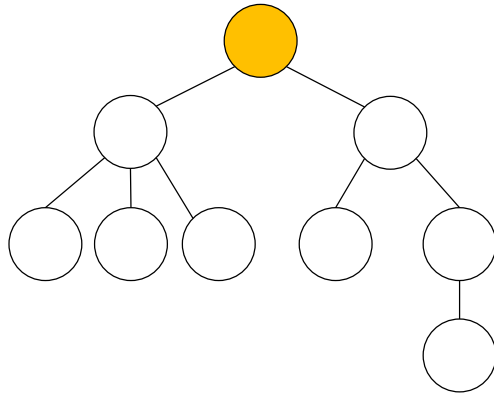


Breadth vs. Depth vs. Best (first search)

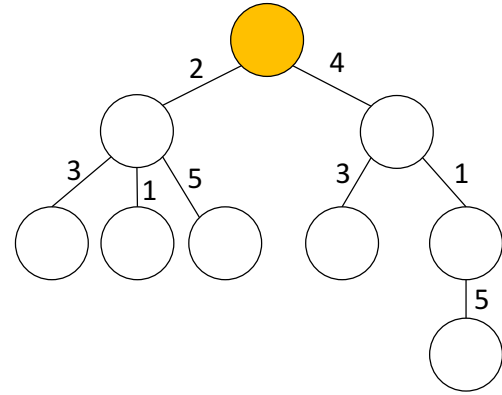
BFS






DFS



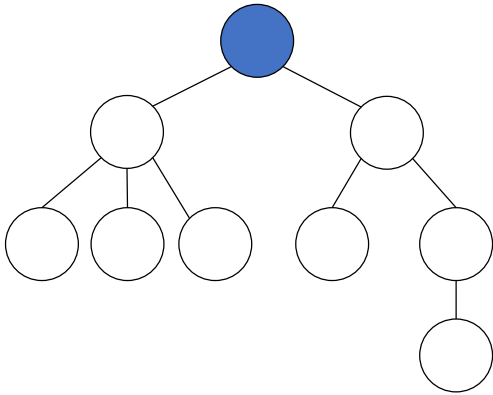
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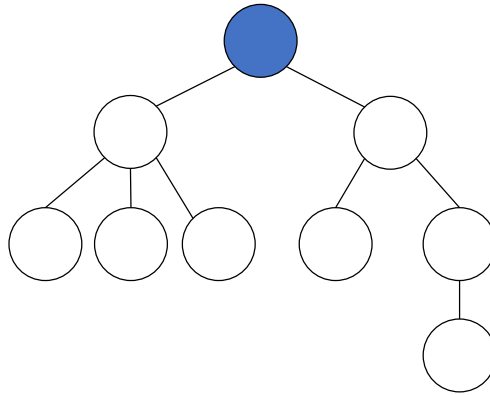
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Breadth vs. Depth vs. Best (first search)

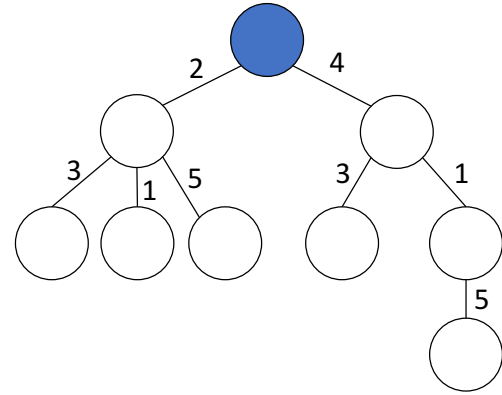
BFS






DFS



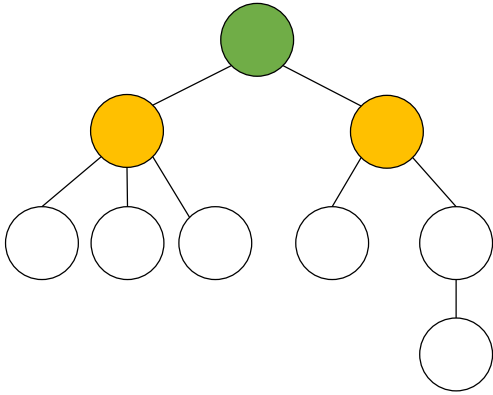
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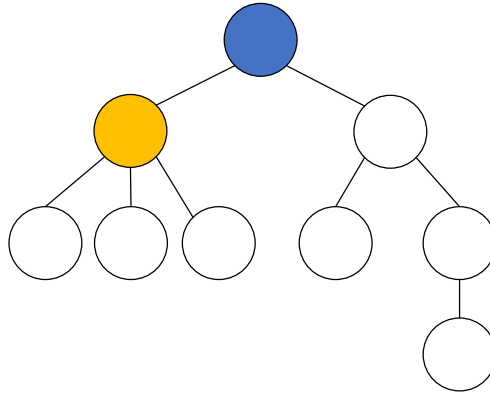
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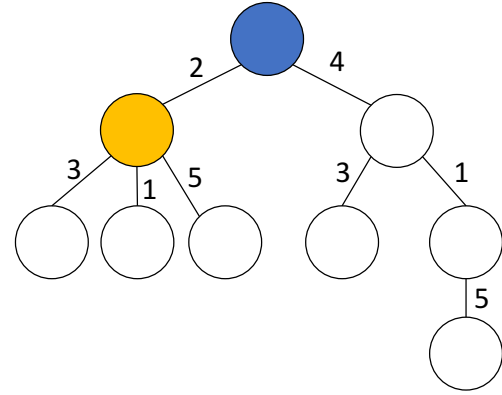
BFS






DFS



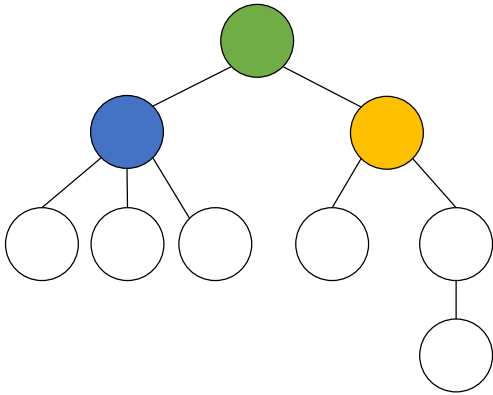
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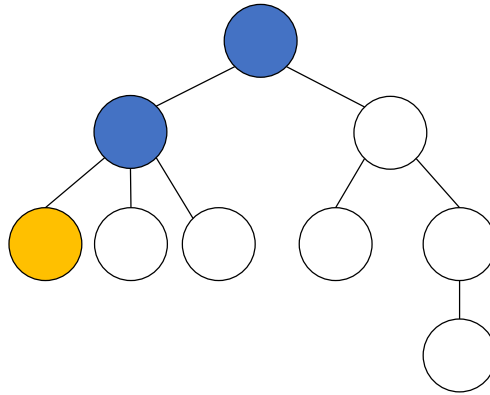
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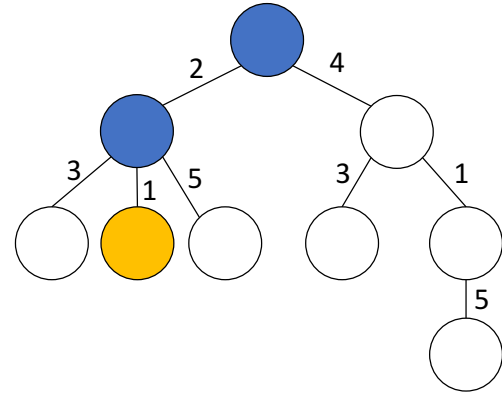
BFS






DFS



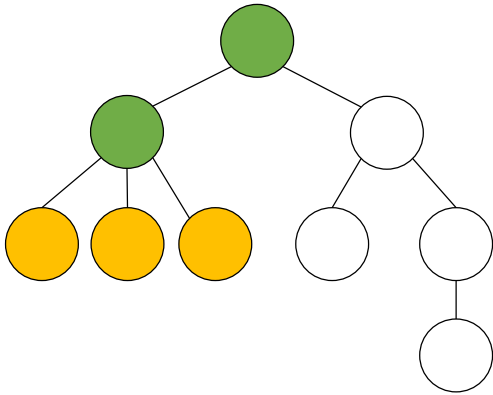
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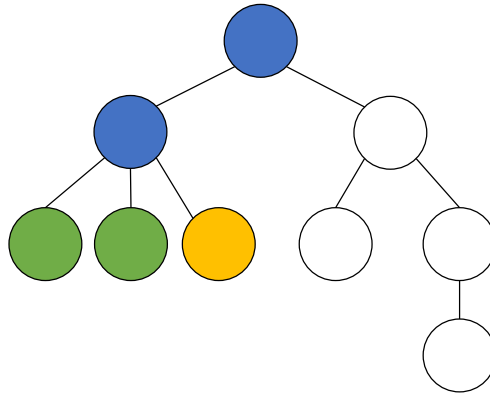
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Breadth vs. Depth vs. Best (first search)

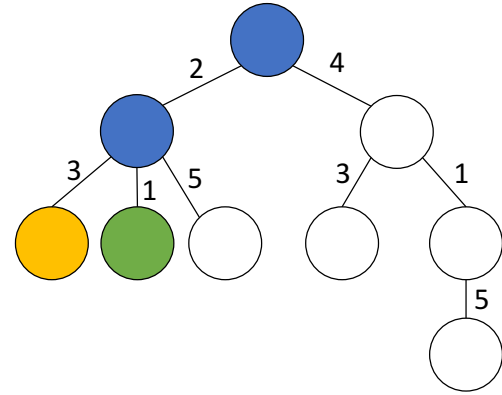
BFS






DFS



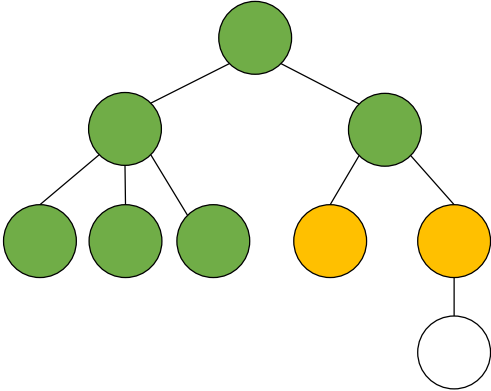
Best



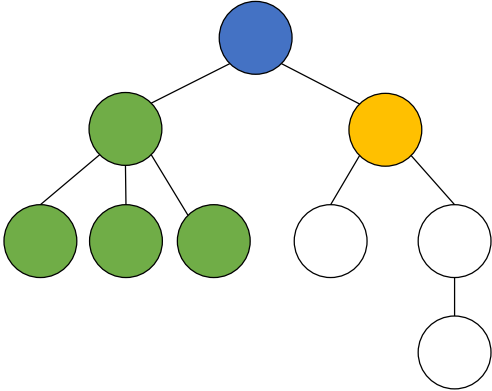
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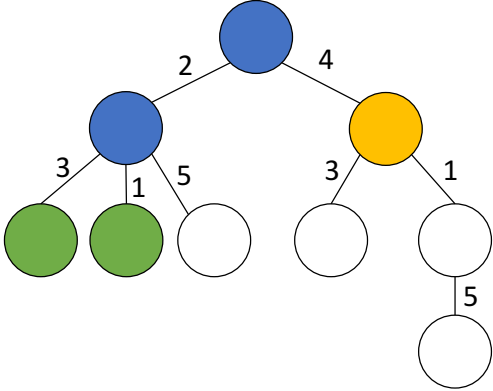
BFS






DFS



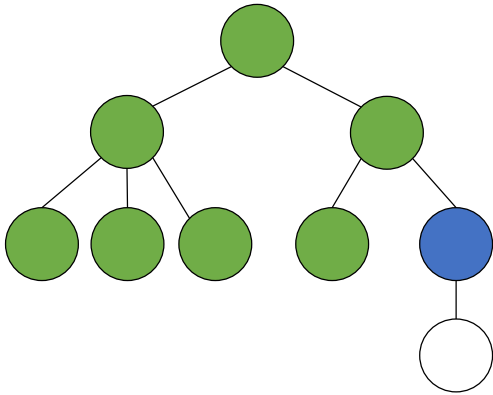
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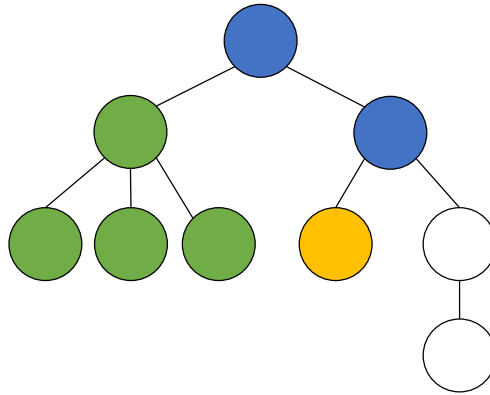
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Breadth vs. Depth vs. Best (first search)

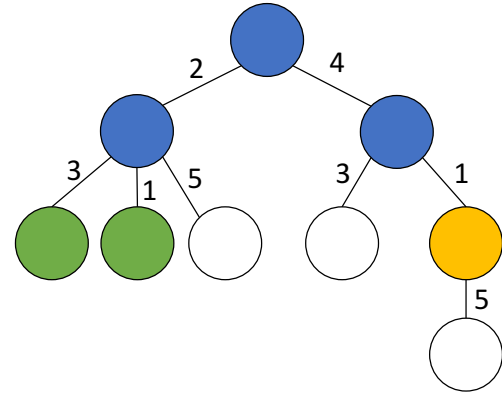
BFS






DFS



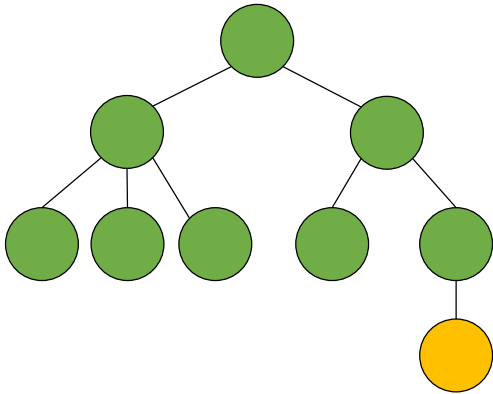
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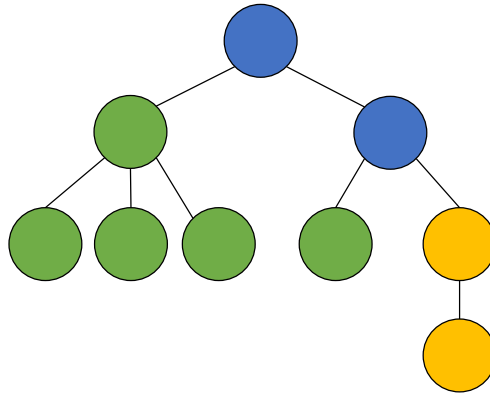
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Breadth vs. Depth vs. Best (first search)

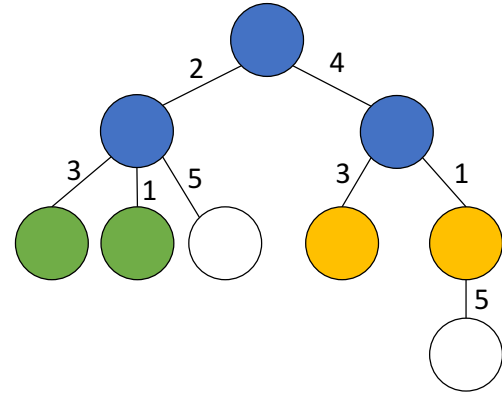
BFS






DFS



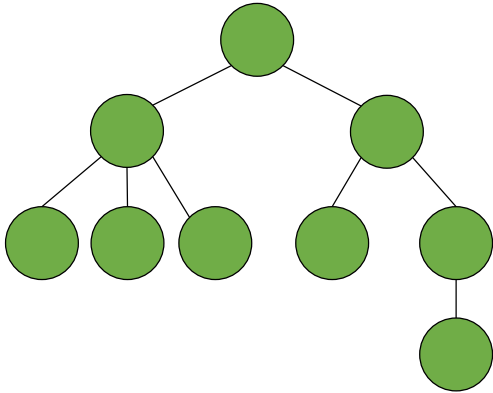
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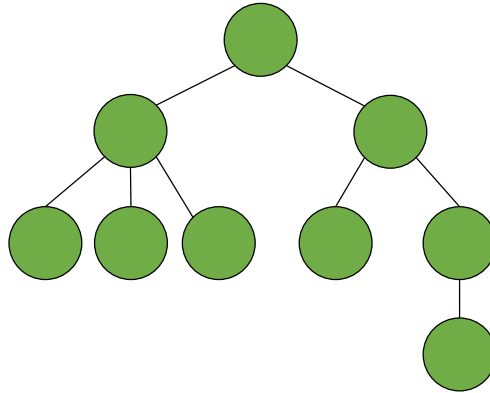
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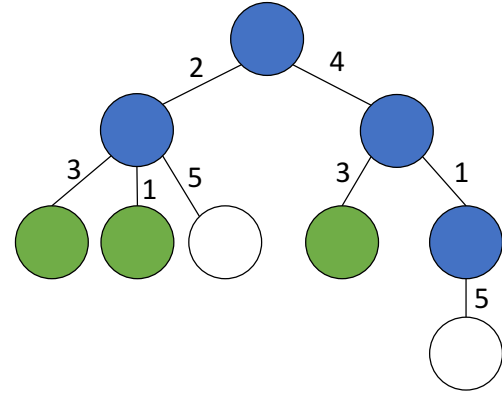
BFS






DFS



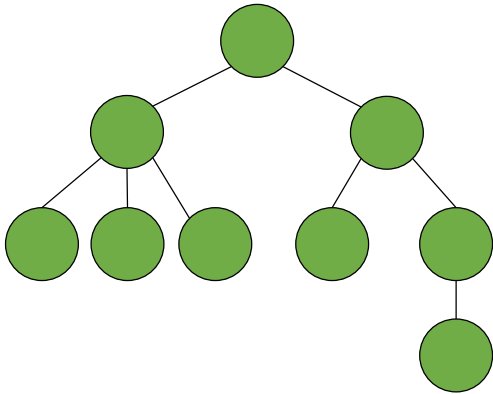
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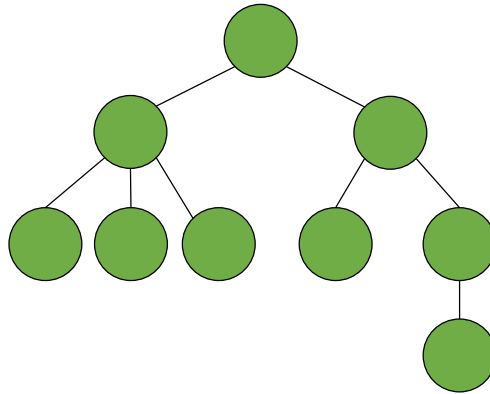
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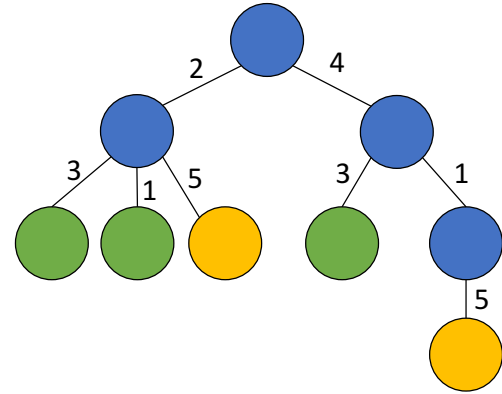
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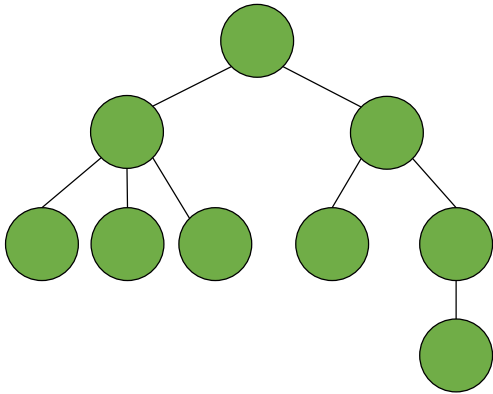
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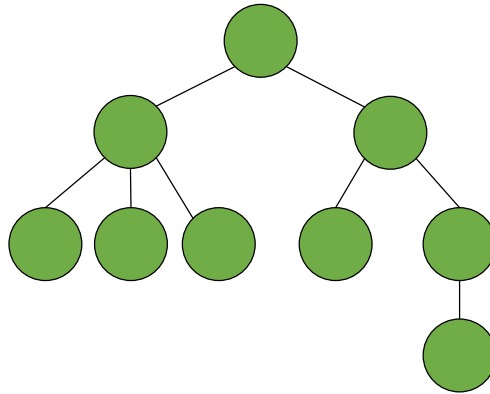
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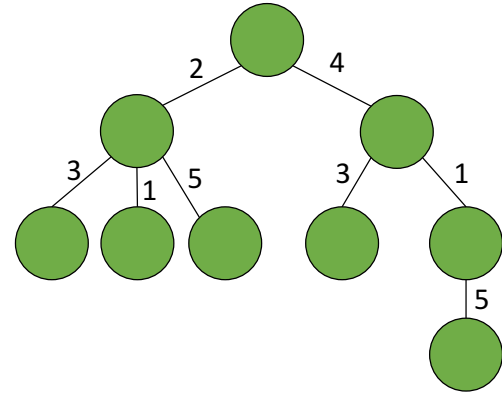
BFS






DFS



Best



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Post-Ordering, DAGs, and Topological Ordering

Post-Ordering

A *post-ordering* of a graph $G = (V, E)$ is an ordering of the nodes based on “when” DFS from each node finished.

To get a post-order, we maintain a global clock variable that is initialized to 1.

Every time we finish calling DFS on all of a node’s neighbors, we set its post-order value to the current value of clock, then increment clock.

Recursive DFS with post-ordering

```
 $G = (V, E)$  is a graph  
visited[ $u$ ] = 0 for all  $u \in V$   
clock = 1
```

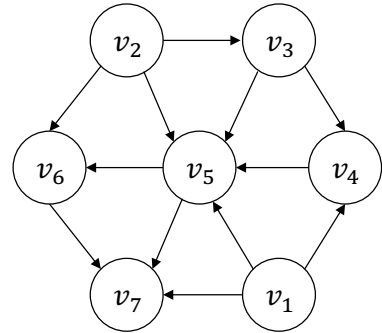
```
DFS( $u$ ):  
    visited[ $u$ ] = 1  
    For  $v \in \text{Neighbors}(u)$ :  
        If visited[ $v$ ] = 0:  
            parent[ $v$ ] =  $u$   
            DFS( $v$ )
```

```
post-visit( $u$ )
```

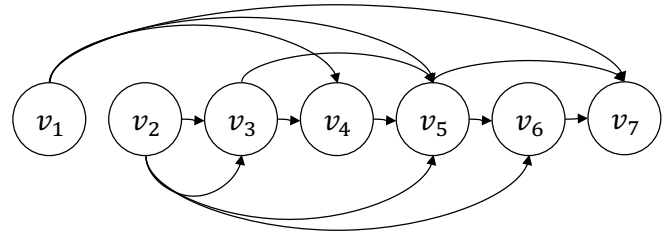
```
post-visit( $u$ ):  
    set postorder[ $u$ ] = clock  
    clock  $\leftarrow$  clock + 1
```

Directed Acyclic Graph (DAG)

- A directed graph with no cycles
- Represent precedence relationships
 - “this” comes before “that”
 - “this” is prior to “that”



A *topological ordering* of a directed graph is a labeling of the nodes so that all edges point “forward”, meaning for all directed edges $(v_i, v_j), j > i$

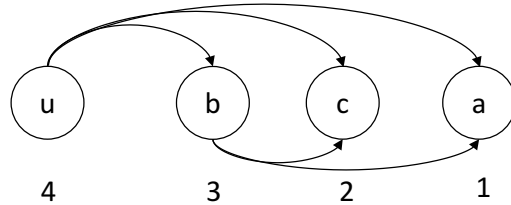
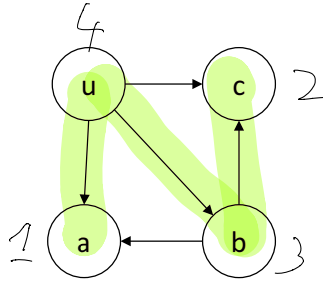


Key point: A reverse post-ordering of the nodes in a DAG is a topological ordering!

Topological Ordering

Ordering nodes by decreasing post-order gives a topological ordering.

Example:



| Vertex | u | a | b | c |
|-----------|---|---|---|---|
| Postorder | 4 | 1 | 3 | 2 |

Minimum Spanning Trees

Minimum Spanning Trees

A tree with n nodes
has $n-1$ edges

A *spanning tree* is a set of edges $T \subseteq E$ is a subgraph of a graph $G = (V, E)$ that (i) is a tree and (ii) contains all of the nodes $v \in V$.

A *minimum spanning tree* for a connected, weighted, undirected graph $G = (V, E, \{w_e\})$, where $w_e \in \mathbb{R}$ is a weight associated with each edge $e \in E$, is a spanning tree T with minimum weight $w(T)$:

$$w(T) = \sum_{e \in T} w_e$$

Borůvka's Algorithm

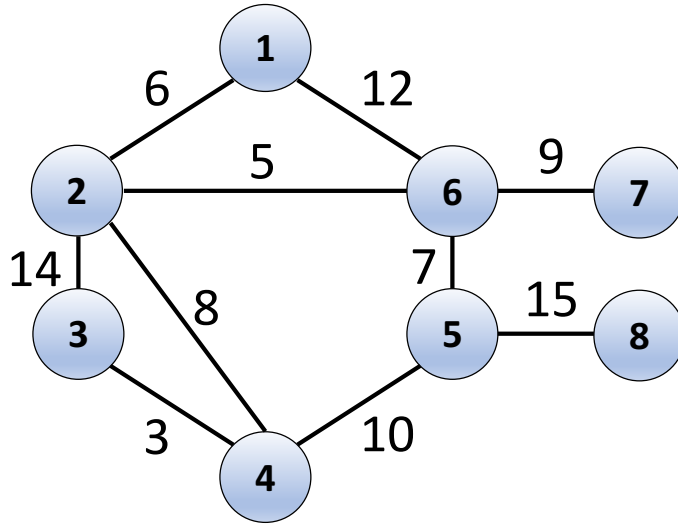
- **Borůvka:**

- Let $T = \emptyset$
- Repeat until T is connected:
 - Let C_1, \dots, C_k be the connected components of (V, T)
 - Let e_1, \dots, e_k be the safe edge for the cuts C_1, \dots, C_m
 - Add e_1, \dots, e_k to T

- **Correctness:** every edge we add is safe

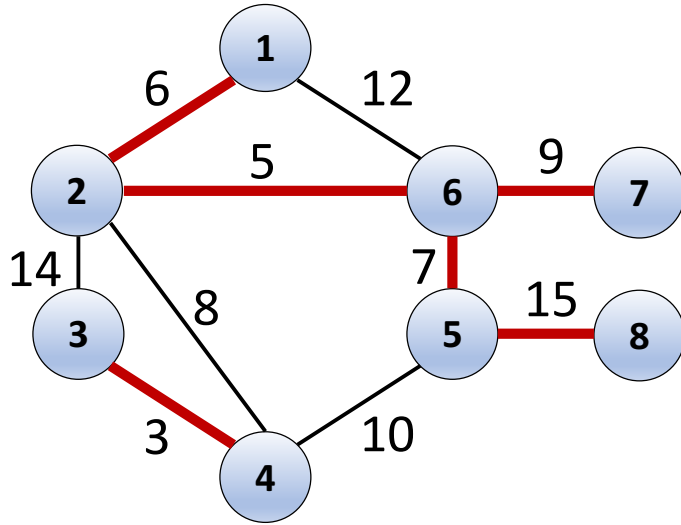
Borůvka's Algorithm

Label Connected Components



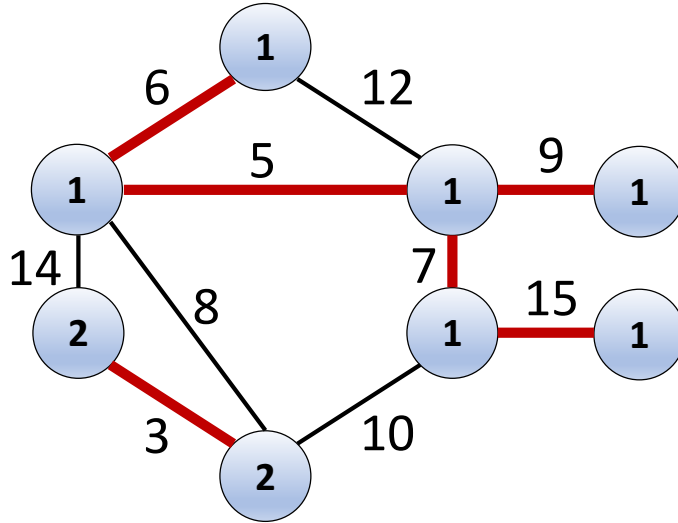
Borůvka's Algorithm

Add Safe Edges



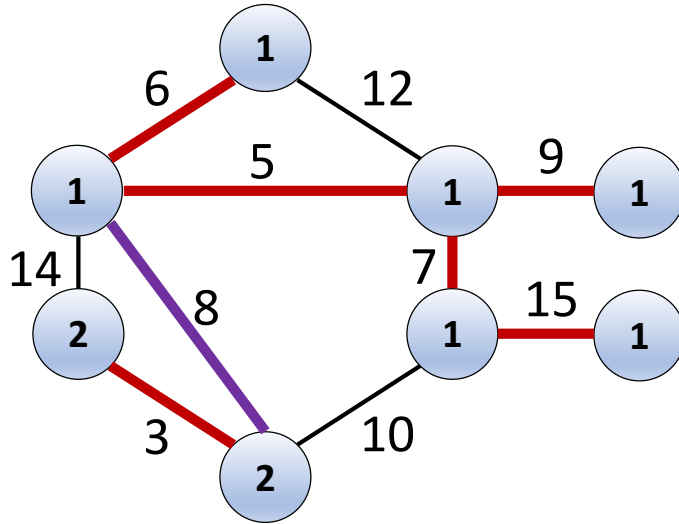
Borůvka's Algorithm

Label Connected Components



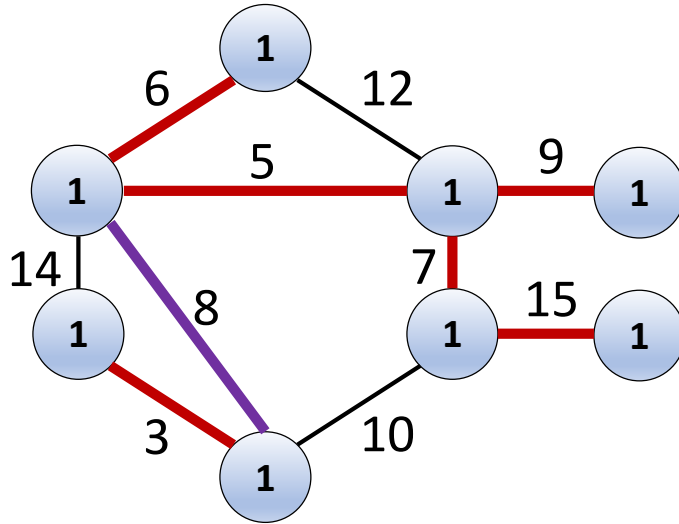
Borůvka's Algorithm

Add Safe Edges



Borůvka's Algorithm

Done!



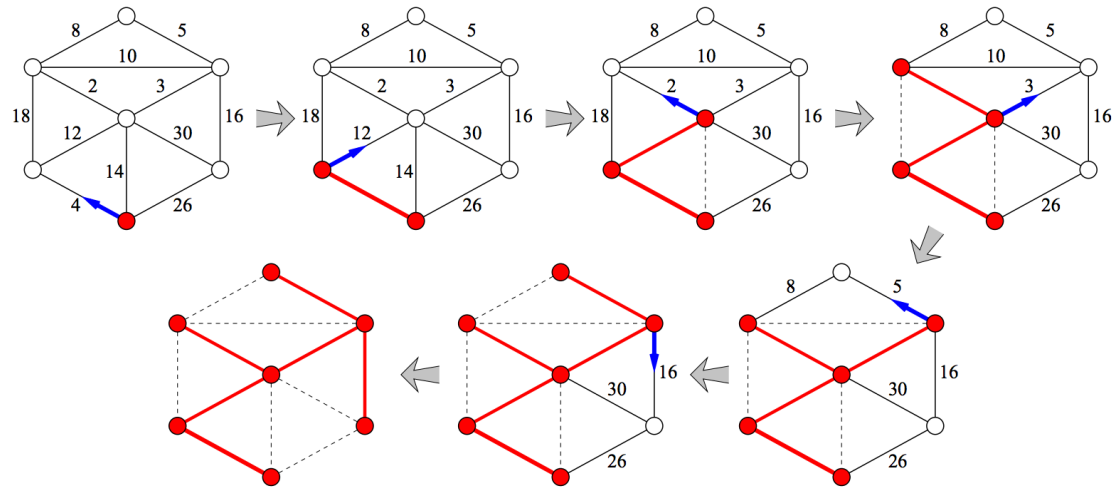
Prim's Algorithm

- **Prim Informal**

- Let $T = \emptyset$
- Let s be some arbitrary node and $S = \{s\}$
- Repeat until $S = V$
 - Find the cheapest edge $e = (u, v)$ cut by S . Add e to T and add v to S

- **Correctness:** every edge we add is safe

Prim's Algorithm



Betweenness Centrality

$$b(u) = \sum_{s \neq t \in V} \frac{\overline{\sigma}_{st}(u)}{\overline{\sigma}_{st}}$$

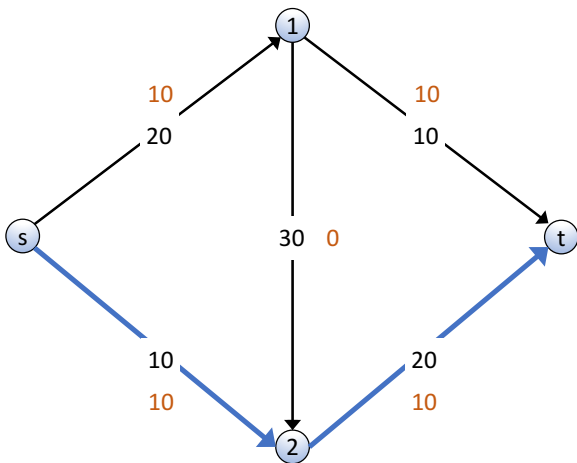
Network Flow

$\overline{\sigma}_{st}(u) \rightarrow$ all shortest paths between s & t that contain u

$\overline{\sigma}_{st} \rightarrow$ all shortest paths between s & t

Augmenting Paths

- Given a network $G = (V, E, s, t, \{c(e)\})$ and a flow f , an **augmenting path** P is an $s \rightarrow t$ path such that $f(e) < c(e)$ for every edge $e \in P$

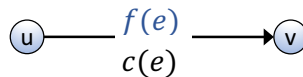


Adding uniform flow on an augmenting path results in a new valid s-t flow!

Residual Graphs

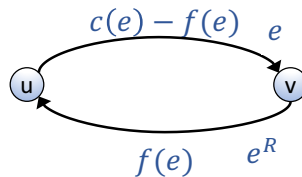
- Original edge: $e = (u, v) \in E$.

- Flow $f(e)$, capacity $c(e)$



- Residual edge

- Allows “undoing” flow
- $e = (u, v)$ and $e^R = (v, u)$.
- Residual capacity



- Residual graph $G_f = (V, E_f)$

- Edges with positive residual capacity.

- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : c(e) > 0\}$.

Augmenting Paths in Residual Graphs

- Let G_f be a **residual graph**
- Let P be an augmenting path in the **residual graph**
- **Fact:** $f' = \text{Augment}(G_f, P)$ is a valid flow

```
Augment( $G_f, P$ )  
   $b \leftarrow$  the minimum capacity of an edge in  $P$   
  for  $e \in P$   
    if  $e \in E$ :    $f(e) \leftarrow f(e) + b$   
    else:         $f(e) \leftarrow f(e) - b$   
  return  $f$ 
```

Note: This is the same process as the recurrence in Erickson 10.3!

$$f'(u \rightarrow v) = \begin{cases} f(u \rightarrow v) + F & \text{if } u \rightarrow v \in P \\ f(u \rightarrow v) - F & \text{if } v \rightarrow u \in P \\ f(u \rightarrow v) & \text{otherwise} \end{cases}$$

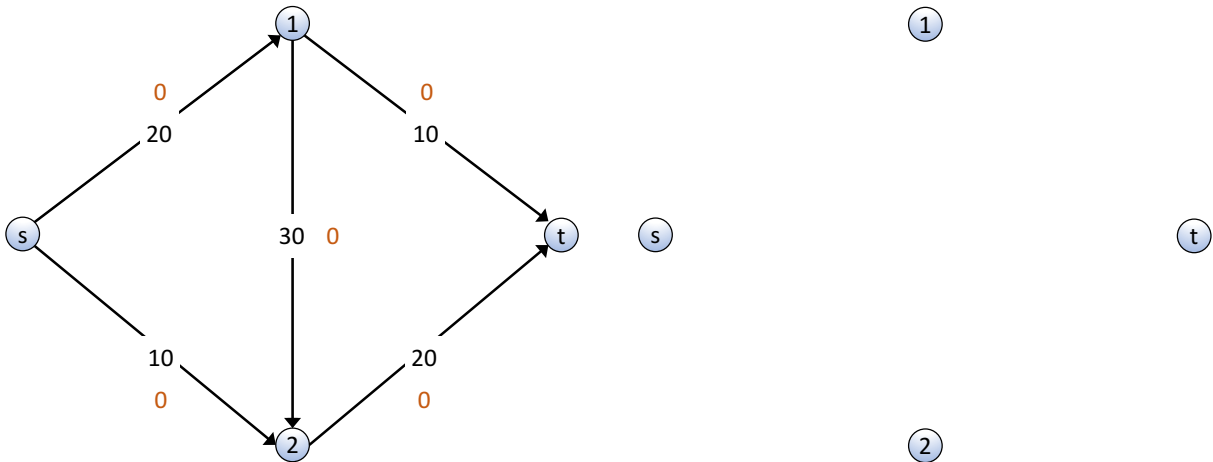
Ford-Fulkerson Algorithm

```
FordFulkerson( $G, s, t, \{c(e)\}$ )  
  for  $e \in E$ :  $f(e) \leftarrow 0$   
   $G_f$  is the residual graph  
  
  while (there is an  $s$ - $t$  path  $P$  in  $G_f$ )  
     $f \leftarrow \text{Augment}(G_f, P)$   
    update  $G_f$   
  
  return  $f$ 
```

```
Augment( $G_f, P$ )  
   $b \leftarrow$  the minimum capacity of an edge in  $P$   
  for  $e \in P$   
    if  $e \in E$ :  $f(e) \leftarrow f(e) + b$   
    else:  $f(e) \leftarrow f(e) - b$   
  return  $f$ 
```

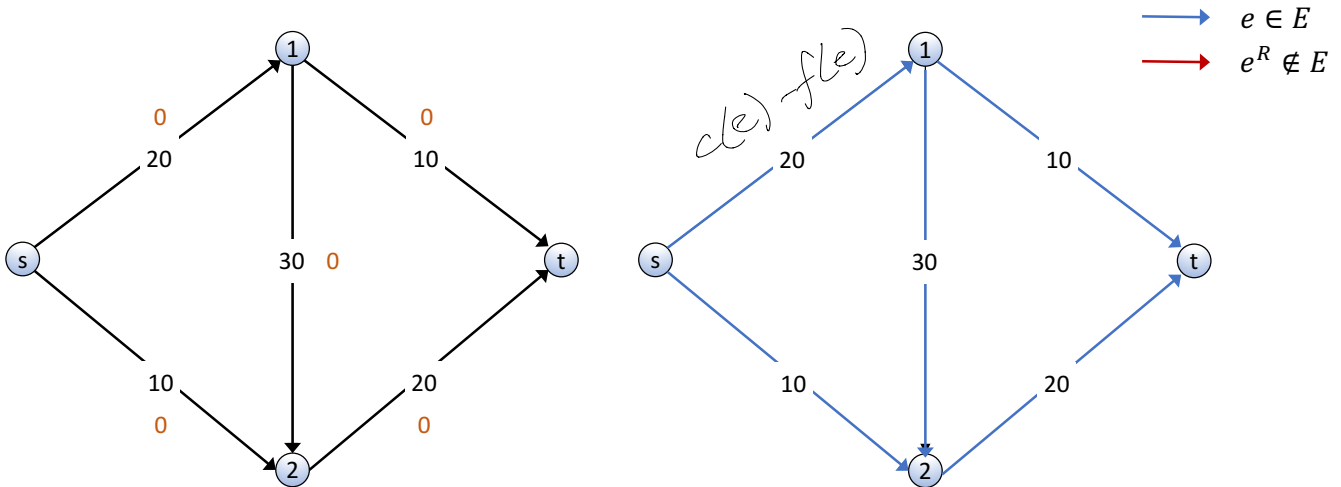
Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
- Repeat until you get stuck



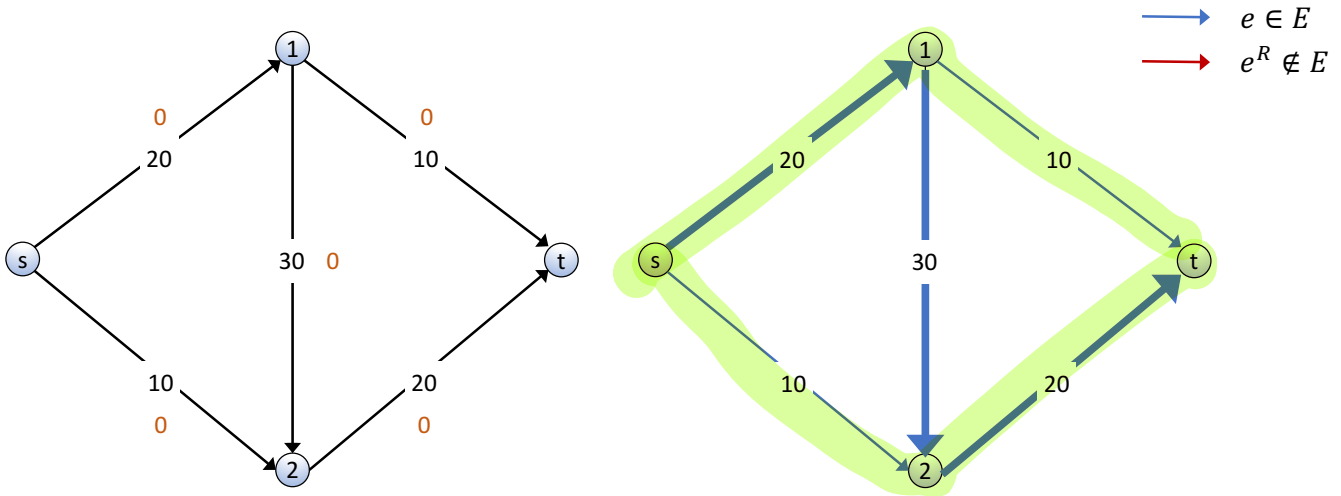
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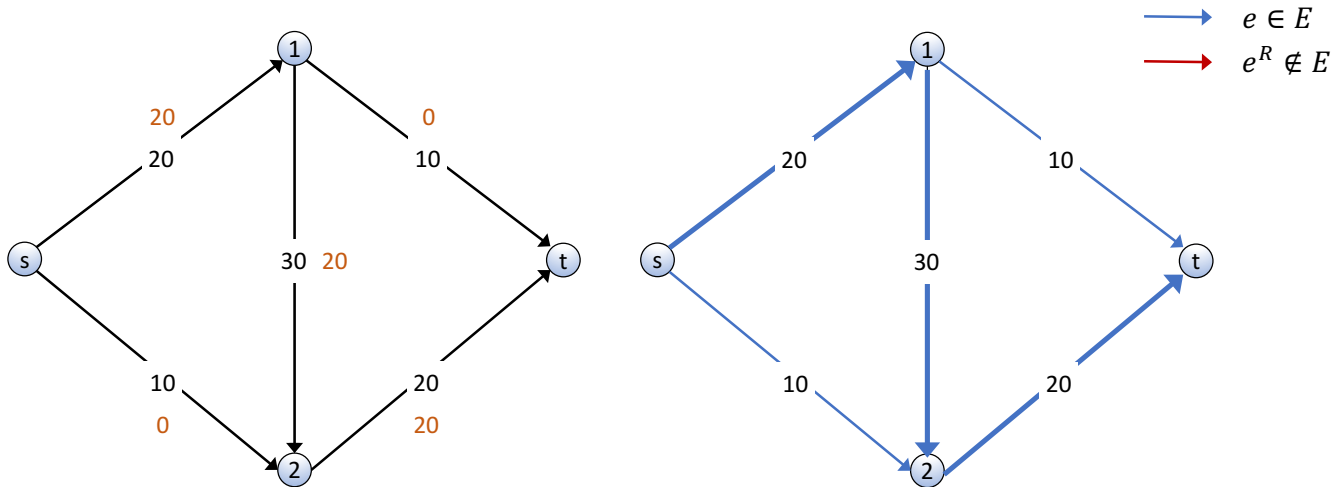
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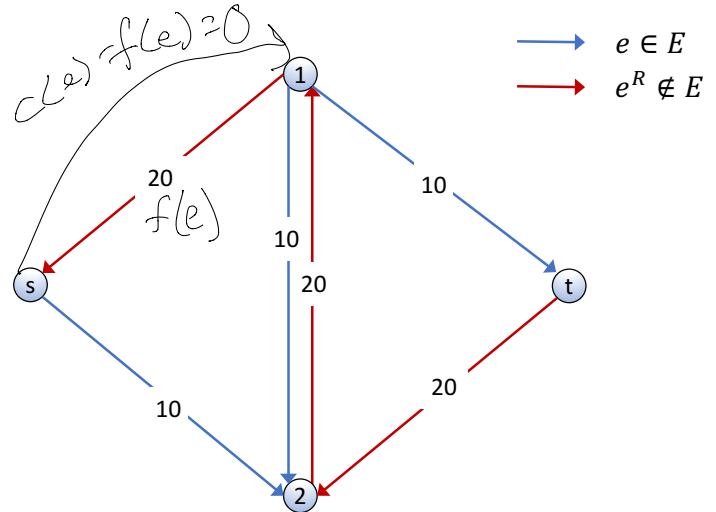
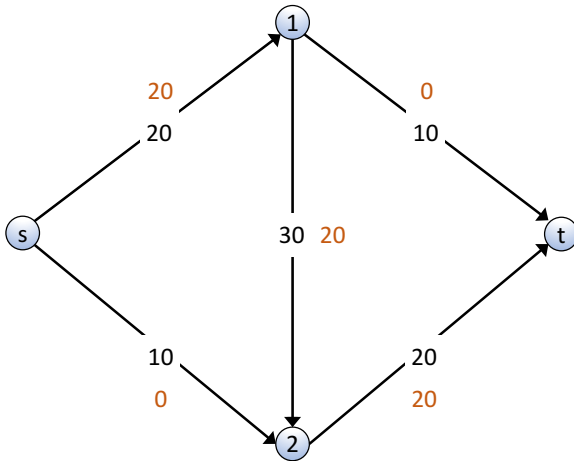
Ford-Fulkerson Algorithm

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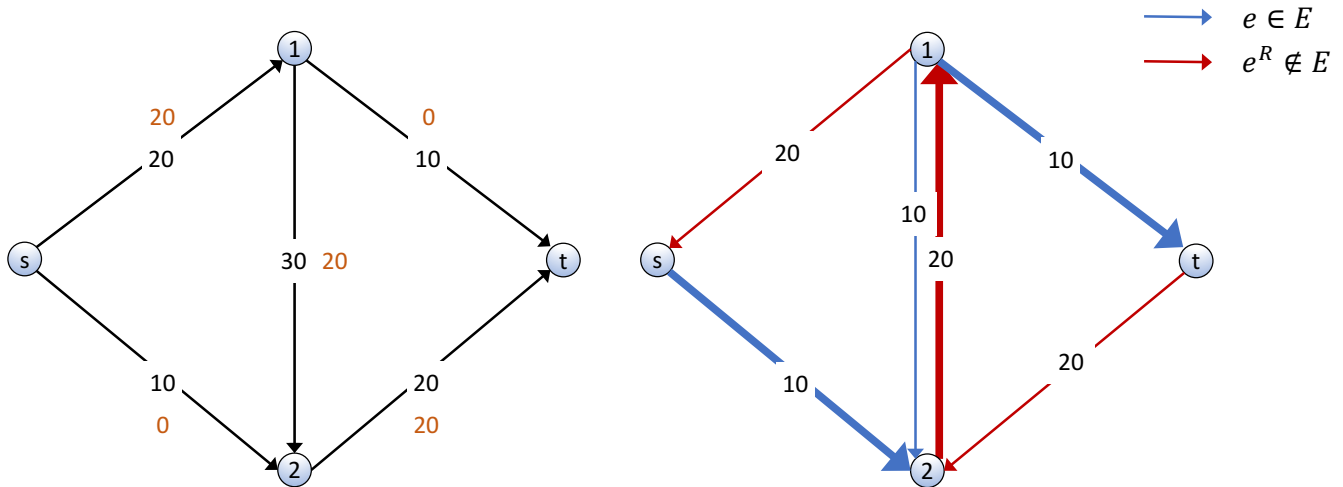
Ford-Fulkerson Algorithm

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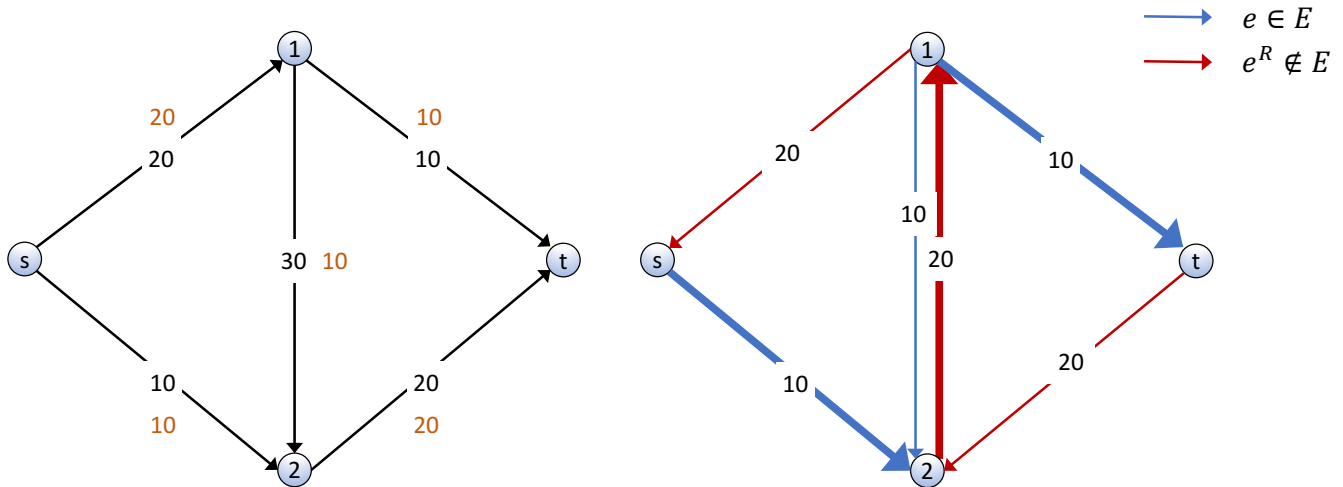
Ford-Fulkerson Algorithm

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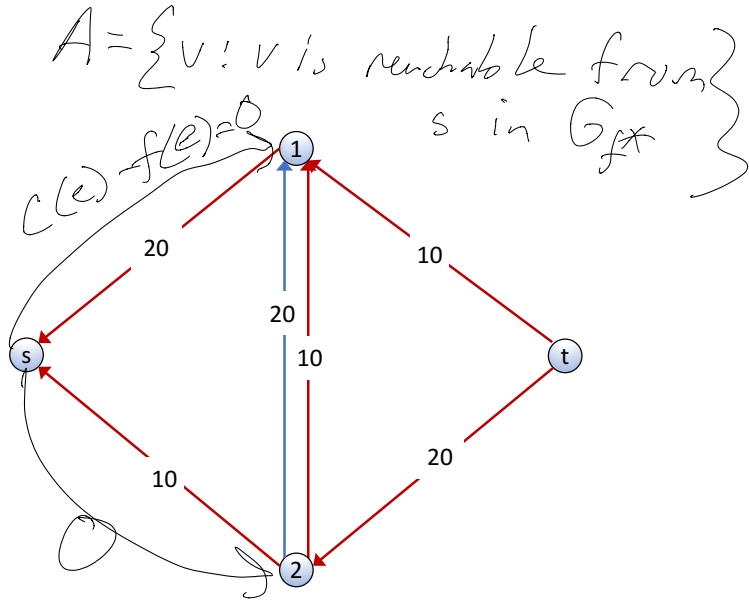
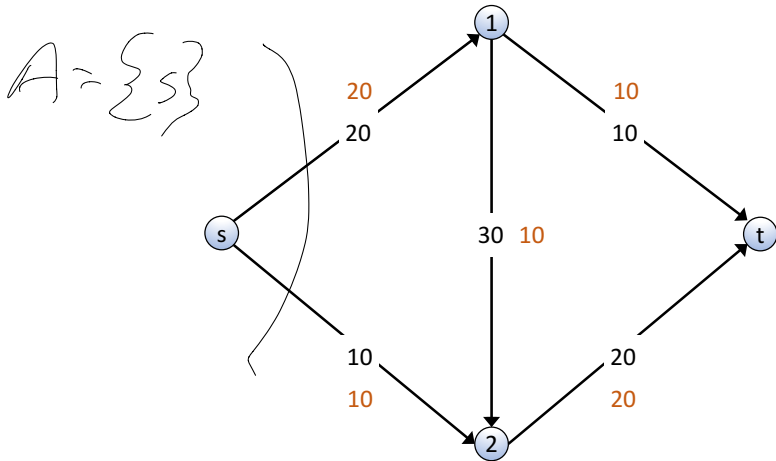
Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$
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Ford-Fulkerson Algorithm

- Start with $f(e) = 0$ for all edges $e \in E$
- Find an **augmenting path** P in the **residual graph**
- Repeat until you get stuck



Network Flow Summary

- **The Ford-Fulkerson Algorithm solves maximum s-t flow**
 - Running time $O(m \cdot val(f^*))$ in networks with integer capacities
- **Strong MaxFlow-MinCut Duality: max flow = min cut**
 - The value of the maximum s-t flow equals the capacity of the minimum s-t cut
 - If f^* is a maximum s-t flow, then the set of nodes reachable from s in G_{f^*} gives a minimum cut
 - Given a max-flow, can find a min-cut in time $O(n + m)$
- **Every graph with integer capacities has an integer maximum flow**
 - Ford-Fulkerson will return an integer maximum flow

More questions?

Wrap-up

No class tomorrow!

Homework 5 due tonight, solutions out tomorrow morning

- Get in touch ASAP (not 10PM) if you need more time!

Midterm 2 released Wednesday 8PM and due Friday 8PM Boston time!